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Applications of Stability Analysis to Nonlinear Discrete Dynamical Systems Modeling Interactions

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A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Abstract

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Many of the phenomena studied in the natural and social sciences are governed by processes which are discrete and nonlinear in nature, while the most highly developed and commonly used mathematical models are linear and continuous. There are significant differences between the discrete and the continuous, the nonlinear and the linear cases, and the development of mathematical models which exhibit the discrete, nonlinear properties occurring in nature and society is critical to future scientific progress. This thesis presents the basic theory of discrete dynamical systems and stability analysis and explores several applications of this theory to nonlinear systems which model interactions involving economic agents and biological populations. In particular we will explore the stability properties of equilibria associated with inter-species and intergenerational population dynamics in biology and market price and agent composition dynamics in economics.

Chapter 1

Introduction

This thesis is concerned with nonlinear discrete dynamical systems and their application to modeling interactions which occur in the natural and social worlds. In particular we are interested in applications to biology involving interactions between species and between different generations of the same species and models involving economic agents making decisions in competitive market environments and adjusting to (or anticipating) the actions of other agents. Due to biological constraints and the sequential nature of human decision making these situations lend themselves to being modeled using discrete dynamical systems, i.e. systems with states which evolve in discrete time steps.

We begin in Part I by presenting the basic theory underlying discrete dynamical systems. In chapter 2 we will present the basic concepts and definitions necessary to understand the nature of discrete dynamical systems. Then, in chapter 3, we will proceed to discuss the notion of system equilibria or "fixed points", specifically we will examine the methods used to determine the local and global stability properties of a given equilibrium and to describe bifurcation phenomena of the equilibria which may occur due to varying system parameters. Applications of this theory will be discussed in Part II and will include some introductory examples and specific cases derived from population modeling in biology in chapter 5 and agents-based modeling and economic theoretical economics in chapter 6. The unifying thread

running through all these examples is the dynamics of interaction. Though all sciences investigates interactions, e.g. the interactions between point masses in mechanics, between molecules in basic chemistry, etc., the interactions which occur at higher level of organization in biological and social systems are exceptionally difficult to predict. This unpredictability is due at least in part to the nonlinearity of such systems and the fact that these subjects also often describe phenomena which occur in discrete time. When systems behave in complex, nonlinear, discrete manners the usual methods of mathematical modeling using differential equations and analysis may not be effective. The theory of nonlinear discrete dynamical systems provides a mathematical toolkit to help understand these complicated situations.

The purpose of this thesis is to help present some of the theoretical and practical results from this still nascent theory and to encourage its further exploration and development.

Part I

Theory

Chapter 2

Dynamic Systems, Discreteness and Nonlinearity

"...mathematical time, of itself, and from its own nature, flows equably without relation to anything external..."∼Isaac Newton [19]

[1]The notion of time provides one of the essential characteristic of dynamical systems. But the "time" that we are interested in for this purpose is not the contemporary idea of physical time, which is subject to myriad qualifications imposed by modern physical theory and experiments, nor the subjectively felt passing of time, but the classical characterization quoted from Newton above, i.e. as a general mathematical parameter. This time parameter serves to index the transitions of the system from one "state" to another. This dependence on time is what makes a system a dynamical one. We will now give a definition of the term "state" used above and the corresponding notion of a "state-space" which provide a framework for discussing dynamic systems.

Definition 1. *The state-space of a dynamical system is a collection of variables depending on time and taking values in a vector space V, such that the information contained in the*

[1]Material and definitions in this section is indebted to [4]

variables at a given time t_0 is enough to determine the values of the variables for all times t. The values of these variables a given time is the state of the dynamical system at that time.

The genesis of the "state-space" framework for representing dynamical systems lies in the philosophy of classical mechanics. It finds its prototypical expression in the following statement from Pierre Simon de Laplace:

> "Given for one instant an intelligence which could comprehend all the forces by which nature is animated *and the respective situation of the beings who compose it*...for it, nothing would be uncertain, and the future, as the past, would be present to its eyes"[17]

Here, within the quote, I have italicized the portion of Laplace's remark that defines in plain language what a "state" is as it stands in the state-space model for systems–the *respective situation* of the *beings* that *compose* nature. In the continuous case covered in classical mechanics the evolution of the system is governed by a set of first-order differential equations, corresponding to Laplace's *animating forces*. If these equations are known together with the system's current state, then we can determine the system's behavior for all time. But, in addition to the continuous states modeled in classical mechanics, the state-space representation lends itself to the modeling and analysis of discrete dynamical systems as well.

In the discrete case, just as in the familiar continuous case, we consider the evolution of a set of state variables depending on time. The critical difference is that time is now taken to be integer valued, and instead of a continuous flow from one state of the system to another governed by differential equations we have discrete jumps from one state to the next. These jumps are governed by "difference equations".

With the assumption that time is divided into discrete steps and takes sequential values from the natural numbers we define the difference equation governing the state variable x by:

$$x(n+1) = f(x(n), n) \qquad x(0) = x_0 \tag{2.1}$$

Where f is a function from the state space to itself. In the case where $f(x(n), n)$ is autonomous, which is to say that it has no explicit dependency on the time variable (here denoted by "n"), the equation (2.1) will be what is called a "recurrence relation" or "recurrence mapping". Regardless of the autonomous or non-autonomous nature of the system we note that while the continuous systems models' transition functions were given as first-order differential equations the analogous difference equation above gives each state as only a function of time and the systems previous state; it does not depend on states farther back in time. This fact is signified by calling such equations "first-order difference equations". Of course, a transition function which depends on states two steps back would be a "second-order difference equation", and those depending on states up to three time steps back are called "third-order" and so on.

The discrete analog to the solution trajectories from ordinary differential equations is given by the sequences of states induced by the mapping for a given initial condition. In the case of a system governed by a recurrence relation this sequence will be given by:

$$\{x_0, f(x_0), f(f(x_0)), ...\}$$

which will be denoted:

$$\{x_0, f(x_0), f^2(x_0)), f^3(x_0)...\}.$$

Each progressive application of the mapping function f to the previous state is called an "iterate" of the system. Naturally the "n^{th}" application of the mapping to the initial condition (i.e. the f^n term in the solution sequence) is called the "n^{th}-iterate" of the system. As the term "solution trajectory through x_0" is applied to the curve generated by the solution to the initial value problem in a continuous dynamic system, this sequence is termed the "positive orbit of x_0" and will be denoted $O(x_0)$ for short.

2.1 Linear Difference Equations

Our focus in this thesis is on *nonlinear* difference equations, but nonlinearity as such is a negative property in that it is defined to be the absence of *linearity*. So to understand it we must first define what it means for a given system to be linear.

Definition 2. *A function $f : D \to R$, where D and R are vector spaces over a field F, is said to be linear if $\forall x, y \in D$, and $\forall \alpha, \beta \in F$ we have*

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

A function is nonlinear if it is not linear.

Nonlinear systems are those which are governed by nonlinear transition functions.

The simplest and most well understood discrete dynamic systems are those which are governed by what are referred to as "affine" equations. In the case of first-order difference equations these will have the form:

$$x(n+1) = a(n)x(n) + b(n), \qquad x(0) = x_0. \tag{2.2}$$

The above $a(n)$ and $b(n)$ take constant values depending on time n with the restriction that $a(0) = 1$, and $b(0) = 0$. In the case where the $b(n)$ terms are identically zero for all n the difference equation can be seen to be governed by a linear function. Some texts (and common convention) will denote *all* equations of the form (2.2) as *linear*, regardless of what values the $b(n)$'s take, however, this is not technically correct, though it is mostly harmless in practice. Throughout this work we will adopt a stricter definition of linearity given above.

The solution to (2.2) will be given by iteration and, following [7], we show it will have the form

$$x(n) = a(n) \cdot a(n-1) \cdots a(2) \cdot a(1) \cdot x_0 = \prod_{i=1}^{n} a(i) \cdot x_0, \tag{2.3}$$

when the $b(n)$ are identically zero for all n i.e. the system is linear.

When the $b(n)$ take values other than zero it will have the form

$$x(n) = \prod_{i=0}^{n-1} a(i) \cdot x_0 + \sum_{k=1}^{n-1} \left(\prod_{j=k+1}^{n-1} a(j) \right) \cdot b(k). \tag{2.4}$$

This solution follows from simple induction. Clearly for $n = 0$ we have $x(0) = a(0) \cdot x_0 + b(0)$, as this merely restates that our initial value for x is x_0. Now if we assume that (2.4) holds for n, then by applying our mapping again we have:

$$x(n+1) = a(n) \cdot \left[\prod_{i=0}^{n-1} a(i) \cdot x_0 + \sum_{k=1}^{n-1} \left(\prod_{j=k+1}^{n-1} a(j) \right) \cdot b(k) \right] + b(n),$$

after distribution and collecting the terms this becomes :

$$\prod_{i=0}^{n} a(i) \cdot x_0 + \sum_{k=1}^{n} \left(\prod_{j=k+1}^{n} a(j) \right) \cdot b(k).$$

This is obviously equivalent to

$$\prod_{i=0}^{(n+1)-1} a(i) \cdot x_0 + \sum_{k=1}^{(n+1)-1} \left(\prod_{j=k+1}^{(n+1)-1} a(j) \right) \cdot b(k).$$

Thus by induction we can conclude that (2.4) holds for all n.

Linear functions are said to display the *superposition* and *homogeneity* principles, the former meaning that the function applied to a sum is equivalent to the sum of the function applied to the individual terms (i.e. $f(x+y) = f(x)+f(y)$), and the latter that the function applied to the additive identity of the domain will again yield the additive identity in the range ($f(0) = 0$). Functions with these properties are special cases and can be said to be the exception as opposed to the rule. The tremendous development of the mathematical theories that exploit the linearity which may be present in a system has been due to the comparative simplicity of linear systems, while the difficulties inherent in the theory of nonlinear systems

have made its progress much slower. This is in spite of the fact that most of the phenomena in the natural world and most mathematical functions themselves are nonlinear.

While linear systems tend to behave predictably, nonlinear systems are much more varied (and potentially deranged) in their behavior. This is greatly exacerbated by the move from continuous to discrete systems. The continuity and differentiability properties of the former are leaned on heavily in the application of approximate linearization techniques to obtain solutions, while in the discrete case, which jettisons continuity and differentiability notions with regard to time, though he state variables themselves may be structured in a continuum, and often the continuity of the transition function in regard to the state variables is assumed. Behaviors that are impossible in the linear case regularly manifest when nonlinearities are introduced, and behaviors in the discrete case manifest that are excluded in the continuous case. Together these facts compound and create behavior, which under the pleasant conditions of linear, continuous analysis, would be incredible in every sense of the word.

2.2 Nonlinear Systems

For comparison's sake, we begin this section by considering the following differential equation:

$$\frac{dx}{dt} = kx - x^2 \qquad x(0) = x_0 \qquad k > 0. \tag{2.5}$$

Here we have a continuous dynamical system with a transition function which only depends on the state variable x and a parameter k. Note that in this case the transition function is nonlinear and that the location of the equilibrium (or the existence of multiple equilibria) depends on the parameter involved. In a single dimension this dependence of the equilibrium location, and the possible existence of multiple equilibria, on the parameters is an emergent property of nonlinear system and is not present in linear cases.

Equation (2.5) as stated will have two equilibrium points, $x = 0$ and $x = k$, These points

partition the real line into three intervals which display differing behavior, for x values left of 0 the states of the system flow leftward indefinitely, between 0 and k the state trajectories move rightward to the equilibrium at $x = k$, while for initial conditions above k the states decay leftward and move towards the $x = k$ equilibrium. This gives evidence to the fact that nonlinear systems can display richer varieties of behavior than their linear counterparts.

The equivalent examples for a nonlinear one-dimensional discrete dynamical system is called the *logistic equation* or *logistic map*. We can proceed from the above continuous example to the discrete case by using Euler's discretization method following [7]. If we wished to approximate (2.5) with a difference equation we could consider a particular domain of interest $[a, b]$ and partition the domain of definition into n intervals of size $h = \frac{b-a}{n}$. Euler's method for differential equations then says that for a given $x \in [a, b]$,

$$\frac{dx}{dt} = kx(t) - x(t)^2 \approx \frac{x(t+h) - x(t)}{h},$$

so that in approximation

$$x(t+h) = (1+kh)x(t) - hx(t)^2.$$

If we consider the discrete time steps generated by incrementing by h to be the domain we will have:

$$x(n+1) = \mu x(n) - \beta x(n)^2, \tag{2.6}$$

where $\mu = 1 + hk$ and $\beta = h$. By making the simplifying change of variables

$$\bar{x}(n) = \frac{\beta}{\mu} x(n)$$

we obtain the system

$$\bar{x}(n+1) = \alpha \cdot \bar{x}(n)(1 - \bar{x}(n)), \tag{2.7}$$

where $\alpha = \frac{\mu^2}{\beta}$.

Assume we are working with a system which can be described by (2.7), where initial values are restricted to the interval $[0, 1]$. Equilibria will occur at $x_0 = 0$ and $x_0 = \frac{\alpha-1}{\alpha}$,, provided that the latter is in fact in within the relevant interval $[0, 1]$. Note that the stability of each equilibrium is dependent on α, and changes as α moves through its possible values. The stability dynamics of equation (2.7) manifest some counterintuitive behavior, a typical trait of nonlinear as opposed to linear difference equations, which we will present following [6]. Consider the set of points which are stationary after two applications of the transition mapping defined in (2.7) i.e. points which satisfy:

$$x(n) = \alpha^2 \cdot x(n)(1 - x(n)) \cdot [(1 - \alpha x(n)(1 - x(n))] .$$

If we treat $x(n)$ as a given variable this equation is quartic, though given that we already know that $x = 0$ and $x = \frac{\mu-1}{\mu}$ are fixed points (and hence generate unintersting period 2 cycles) we can divide out the resulting quartic by these factors, ending up with a quadratic equation. The roots of this quadratic equations will be given by:

$$\lambda = \frac{(1 + \alpha) \pm \sqrt{(\alpha - 3)(\alpha + 1)}}{2\alpha}.$$

This implies that period 2 cycles will only exist if $\alpha \geq 0$. It is also known that period 4 cycles will only manifest when α reaches $1+\sqrt{6}$. Likewise higher and higher degree 2^n cycles will appear more and more quickly as α increases, and then can "collapse" to cycles of smaller degree, only to explode again. These chaotic dynamics do not appear in continuous one-dimensional systems, let alone linear ones. For example simple partial fraction decomposition lets us solve the continuous case of the logistics equation to see that has the solution

$$x(t) = \frac{1}{1 + Ae^{-kt}},$$

which has comparatively sedate dynamics.

Chapter 3

Equilibria and Stability Analysis of Nonlinear Discrete Dynamical Systems

We now present the methods and tools used for the analysis of nonlinear discrete dynamical systems. Specifically, we will extend or amend the notions involved in the first-order, one-dimensional equations, which we briefly discussed in the previous chapters, to those of higher order and dimension.

To begin, a brief restatement of the general form of a first-order difference equation (2.1):

$$x(n+1) = f(x(n), n) \qquad x(0) = x_0.$$

For notational compactness from here on out we will often use subscripts to indicate the integer dependence of the equation's terms, so that the above (2.2) would appear as:

$$x_{n+1} = f(x_n, n) \qquad x(0) = x_0.$$

The analysis of dynamical systems is primarily concerned with the equilibria of systems and the various properties of those equilibria. A formal definition of what constitutes an equilibrium of a system follows:

Definition 3. *Consider the system given by the transition equation (2.1) where* $f : D \to R^n$, $D \subset R^n$. *An element* x^* *in the domain of* f *is said to be an equilibrium of the system if* $f(x^*) = x^*$.

In the continuous case, autonomous dynamical systems that reach equilibrium must have started at that equilibrium: i.e if $\frac{dx}{dt} = f(x,t)$ with f being continuous and $f(x,t_k) = 0$ for some $t_k > 0$ then it must be the case that $f(x,t) = 0$ for all t in the relevant time domain. However, in a discrete dynamical system it is possible for the state variable to reach an equilibrium state in a finite number of iterations. Systems which exhibit sequences of state variables which begin off an equilibrium and end up constant at that equilibrium are said to have *eventually fixed points.*

Just as in the continuous case, equilibria can be categorized based on their behavior and the primary motive for our analysis is to determine whether or not a given equilibrium is *stable.* In order to define the term *stable* we need to introduce the concept of a *norm* on a vector space:

Definition 4. *Let* V *be a vector space. A norm on* V *is a function* $\|\cdot\| : V \to R$ *such that the following hold*

1. $0 \leq \|v\| \quad \forall v \in V$
2. $\|v\| = 0 \iff v = 0$
3. $\|\alpha v\| = |\alpha|\|v\| \quad \forall v \in V, \alpha \in F$
   (where F *is the field over which* V *is a vector space.)*
4. $\|v + u\| \leq \|v\| + \|u\| \quad \forall u, v \in V.$

Essentially a norm establishes a partial order among the vectors in a vector space analogous to the role played by the absolute value on the real line. Using this order we can establish notions of sequential convergence and hence of stability:

Definition 5. *Let x^* be an equilibrium of a system defined as in (2.1). x^* is called stable if for any given $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x^* - x_0\| < \delta$ implies that $\|f^n(x_0) - x^*\| < \epsilon$ for $n > 0$. If the equilibrium is not stable it is called unstable.*

It should be noted that it is possible for a point arbitrarily close to x^* to generate an orbit which stays arbitrarily close to x^* in the above sense, but which does not tend to converge towards x^*. For instance, the orbit could merely circle around the equilibrium staying within the prescribed bounded region but not approaching the equilibrium. When initial conditions that are sufficiently close to x^* eventually approach x^* we call the equilibrium (or an invariant set of points exhibiting the same "attractive" property) an *attractor*. Formally this notion of an *attractor* can be stated as follows.

Definition 6. *Let A be an subset of a vector space V such that A is invariant under the transition function $f : V \to V$ (i.e. $f(A) \subseteq A$). Define the distance between A and a point x, $\rho(A, x)$ as $\rho(A, x) = \min_{a \in A} \|x - a\|$. Then if there exists an $\epsilon > 0$ such that $\rho(A, x) < \epsilon$ implies $\lim_{n \to \infty} \rho(A, f^n(x)) = 0$ then we say A is an attractor for the dynamical system with governing equation f.*

An equilibrium which is also a stable attractor is called an *asymptotically stable* equilibrium. Since continuous functions are much less prone to exhibit unpredictable behavior than discrete functions much of our analysis exploits continuous features that may be present in discrete systems. Hence if $f : D \to R$, $D \subseteq R$, is continuously differentiable with respect to the state variable x (i.e. $f \in C^\infty(D)$, $D \subseteq V$, where V is the state space) and the system given by

$$x_{n+1} = f(x(n)) \qquad x(0) = x_0$$

has an equilibrium point x^* then we have that $|f'(x^*)| < 1$ implies x^* is stable while $|f'(x^*)| > 1$ implies that x^* is not stable. Cases like these where $|f'(x^*)| \neq 1$ are called hyperbolic. Further analysis for the case where $|f'(x^*)| = 1$ relies on liberal application of Taylor's Theorem to use the fact that $f(x(n))$ is continuously differentiable to eventually determine the behavior of nearby points [10]. Deeper analysis of higher order terms is necessary to determine stability properties. As these techniques can become rather specialized and particular to one dimensional systems, we will neglect further elaboration along these lines. Readers are encouraged to consult the bibliography, particularly [10, 11, 13], for more detail about this particular area of analysis.

3.1 Higher Dimensional Difference Equations

Our purpose in presenting the theory of discrete dynamical systems is to lay a foundation to explore some particular applications of that theory to the biological and social sciences which model interaction. Since the term "interaction" implies systems with two or more state variables we will now investigate the broader extension of the above one-dimensional methods to higher order systems, this will allow us to examine the particular cases of interest. As our application examples involve two-dimensional systems we will present the theory explicitly in the two-dimensional case.

Consider the discrete dynamical system governed by the transition equations:

$$\begin{aligned} x_1(n+1) &= f_1(x_1(n), x_2(n)) \\ x_2(n+1) &= f_2(x_1(n), x_2(n)) \end{aligned}, \tag{3.1}$$

where f_1, f_2 are functions from the relevant domain to the relevant range of interest (in this case $D \subseteq R^2$ and R^2, respectively). Consider the case where f_1 and f_2 are linear, then the

above system could be written in the form

$$x_{n+1} = Ax_n,$$

where the state variable is now a vector in R^2 and A is a 2×2 matrix with constant coefficients. The stability of a fixed point x^* in such a system is characterized by the eigenvalues of the matrix associated with the system. If the maximum modulus for the associated complex eigenvalues (λ_1,λ_2) is greater than one then the equilibrium is unstable, if it is less than one then the equilibrium is stable, and if it is exactly one and the matrix has a single eigenvalue (i.e. $\lambda_1 = \lambda_2 = 1$) then the equilibrium is unstable [6].

The most fundamental tool involved in the analysis of nonlinear discrete dynamical systems is "linearization" of the system about an equilibrium point. In a one-dimensional system given by the differentiable transition equation $f : D \to R, D \subseteq R$ this linearization near the equilibrium point x^* takes the form

$$f(x) \approx f^{'}(x^*)(x - x^*) + x^*.$$

For two-dimensional discrete dynamical systems with continuous differentiable transition functions f_1 and f_2 the "linearization" associated with the system given by

$$\begin{aligned} x_1(n+1) &= f_1(x_1(n), x_2(n)) \\ x_2(n+1) &= f_2(x_1(n), x_2(n)) \end{aligned}.$$

is determined by the Jacobian matrix of the system:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{x_2} \end{bmatrix}$$

To fully explain how this local dependency on the systems linearization of the equi-

librium's stability properties arises we will introduce one of the other significant tools in nonlinear analysis, Liapunov Theory.

3.2 Liapunov Theory

The basis of Liapunov Theory[1] is the *Liapunov Function* associated with a given system of difference equations. Assume that we have a vector $x \in R^m, m \in N$ and the transition function for the system is given by (2.1) with $f : D \to R^m, D \subseteq R^m$ and is autonomous. For a given functional V from $R^m \to R$ we define the variation of V as

$$\triangle V(n) = V(x(n+1)) - V(x(n)).$$

The definition of Liapunov function for discrete dynamical systems is analogous to the definition from the continuous case, with the variation serving the role in the discrete case that the derivative does for continuous systems.
We now follow [6] and define a Lyapunov function by Definition 7.

Definition 7. *A function V from an open subset G of R^m into R is a Liapunov Function on the set G if:*

1. *V is continuous on G*

2. *$\triangle V \leq 0$ whenever both $x(n)$ and $x(n+1) = f(x(n))$ are in G.*

The Liapunov function V is said to be *positive definite* at the fixed point x^* if there exists an open ball $B_\epsilon(x^*)$ centered at x^* such that $V(x) > 0$ for all $x \in B_\epsilon(x^*)$. If the same case holds but with $V(x) < 0$ then V is called *negative definite*. We are now prepared to state a series of theorems which allow us to determine the stability of a given equilibrium in R^2 using Liapunov functions. Our presentation of these theorems follows [6] and [20]

[1]Material in this section is derived from [6],[7] [18], and[20]

Theorem 1. *If there exists V a positive definite Liapunov function on the open ball $B_\epsilon(x^*)$ where x^* is the fixed point of an m-dimension discrete dynamical system with continuous transition function, then we can conclude that x^* is stable. If the variation of V under the mapping is negative on $B_\epsilon(x^*)$ then we can also conclude that x^* is asymptotically stable. If this holds true when $B_\epsilon(x^*)$ is extended to all of R^m and $V(x)$ goes to infinity as $|x|$ goes to infinity, then x^* is globally asymptotically stable.*

Notice that this result allows us to a conclusion regarding the *global* behavior of a system if certain conditions are met. This is a very powerful conclusion to be able to draw. Information contained in the linearization of a system generally can only tell us about local behavior around the fixed point about which we are linearizing. Global results are normally much more difficult to obtain. However, local results are good to have and often are satisfactory for given applications. We may now return to examine the process of linearization and use our new theorem to study the behavior of systems about a given equilibrium.

Recall from the previous section that the linearization of a given 2-dimensional autonomous discrete dynamical system with sufficiently smooth governing transition function f is given by the system of equations:

$$x_{n+1} = A \cdot x_n$$

Where here x_n and x_{n+1} are 2×1 vectors real valued vectors and A is a 2×2 matrix given by:

$$\mathrm{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{x_2} \end{bmatrix}$$

We now introduce a theorem which serves for the discrete case the role that the Hartman-Grobman Theoreom does for continuous systems.

Theorem 2. *Let $f : D \to R^m, D \subseteq R^m$ be continuous, differentiable map with regard to*

all system state variables defined on an open subset around a fixed point x^* and let A be the Jacobian matrix of the system about x^*. Then:

1. *If the maximum modulus of the eigenvalues of A is less than one then x^* is asymptotically stable.*

2. *If the maximum modulus of the eigenvalues of A is greater than one then x^* is unstable.*

3. *If the maximum modulus of the eigenvalues of A is equal to one then no conclusion is drawn.*

Proof of this theorem rests on particular properties of Liapunov functions, thus we will construct an explicit Liapunov function for our system. We will do so by defining a quadratic form on our R^2 space. If x is a 2×1 vector of state variables, let $B = (b_{ij})$ be a real symmetric 2×2 matrix. We will define V: $R^2 \to R$ by

$$V(x) = x^T Bx,$$

where the superscript T indicates the transpose operation. Given our restrictions this equation will become

$$V(x) = b_{11}x_1^2 + 2b_d x_1 x_2 + b_{22}x_2^2,$$

where b_d is equal to the off diagonal elements (which are identical due to the imposed symmetry condition). We wish to show that $V(x)$ as just defined is indeed a Liapunov function when we let B be a positive definite matrix. Clearly the continuity condition is satisfied, thus we only need to check whether or not $\triangle V \leq 0$ whenever both $\hat{x}(n)$ and $\hat{x}(n+1) = f(\hat{x}(n))$ are in G, where G is defined to be some open subset of R^2 containing the equilibrium. Assume that we have chosen our domain such that both $\hat{x}(n)$ and $\hat{x}(n+1) = f(\hat{x}(n))$ remain in it for suitable n. We can investigate the variation of $V(x)$

$$\triangle V(x(n) = x^T(n+1)Bx(n+1) - x^T(n)Bx(n)$$

Since we are working in the linearization of the system we can write the above as:

$$\triangle V(x(n)) = x^T(n)A^TBAx(n) - x^T(n)Bx(n),$$

which simple factoring reveals to be

$$\triangle V(x(n)) = x^T(n)(A^TBA - B)x(n).$$

Thus the second condition will be satisfied if and only if $A^TBA - B$ is equivalent to some $-C$ where C is positive definite. Two results regarding the maximum eigenvalues which will assist us in proving Theorem 2 follow (results derived from [20]):

Lemma 1. *1. The maximum modulus of the eigenvalues associated with A will be less than one if and only if for any given positive definite C, the equation $B - A^TBA = C$ has a unique solution B which is also symmetric and positive definite.*

2. If the maximum modulus of the eigenvalues associated with A is greater than one then there exists B such that given a positive definite C, $B - A^TBA = C$ holds but B is not positive semidefinite.

If we are in the linear case, i.e if we are working with an equation of the form:

$$x_{n+1} = Mx_n \qquad x(0) = x_0, \tag{3.2}$$

with M a matrix of constant terms and x a vector we can see that Theorem 2 holds directly, since we know the solution of (3.2) will be:

$$x_n = M^n x(0). \tag{3.3}$$

To prove stability it is sufficient to show that

$$\lim_{n\to\infty} M^n = 0.$$

To see this, let J be the Jordan form of M, then $M^n \to 0$ only if each Jordan block of J is such that $J^n \to 0$, clearly this will be the case if J is diagonal and the maximum modulus eigenvalue if M is less than one.[2] Also this will fail if the maximum modulus eigenvalue of M is greater than one. With this information we can prove the two statements of our lemma above.

Proof. Lemma 1

Consider the linear system given by

$$x_{n+1} = Ax_n \qquad x(0) = x_0,$$

where A is a constant matrix. Assume that the maximum modulus of the eigenvalues of A is say $\rho < 1$, In [20] it is proven that there exists a nonsingular matrix Q such that for $\epsilon > 0$, we have that in the $\|\cdot\|_2$ norm $\|Q^{-1}AQ\|_2 \leq \|\rho\|_2 + \epsilon$. Thus we have a nonsingular Q with $\|Q^{-1}AQ\|_2 < 1$. Let $B = Q^TQ$. We then have that

$$x^T(B - A^TBA)x = x^TQ^TQx - x^TA^TQ^TQAx$$

$$= \|Qx\|_2^2 - \|QA(Q^{-1}Q)x\|_2^2$$

$$\geq \|Qx\|_2^2 \cdot (1 - \|QAQ^{-1}\|_2) > 0,$$

so we can conclude that $B - A^TBA$ is positive definite. To prove the other direction we introduce a new norm $\|\cdot\|_M$ for a given positive definite matrix M defined so that for a given vector x, $\|x\|_M = x^TMx$. Clearly from the positive definiteness of M we see that this is in

[2]Even if the maximum modulus eigenvalue is one, but the M is "spectral radius diagonalizable" i.e. all of the maximum modulus eigenvalues lie on diagonal Jordan blocks, this will still hold.[6]

fact a norm. Now given our above A and assuming there is a unitary positive definite matrix B such that $B - A^T BA$ we have that for a given vector x

$$\|Ax\|_B^2 = x^T A^T BAx,$$

and since $B - A^T BA$ positive definite so we know that for all x

$$x^T Bx - x^T A^T BAx > 0$$

so that

$$\|x\|_B^2 = x^T Bx > x^T A^T BAx = \|Ax\|_B^2.$$

Since this holds for all x all of the eigenvalues of A must be less than one. □

Now we are equipped to prove Theorem 2 for the general, nonlinear case.

Proof. Theorem 2

We are interested in solutions to the difference equation

$$x(n+1) = f(x(n)) \qquad x(0) = x^*,$$

where x^* is the equilibrium for the defined system that we are interested in and f is considered sufficiently smooth in the state-space variables. From the Taylor's Theorem we know that

$$f(x) = f(x^*) + A(x - x^*) + g(x, x^*),$$

where A is the Jacobian matrix for the system evaluated at the equilibrium and $g(x, x^*) \to 0$ as $x \to x^*$. Letting $H = x - x^*$ we then have

$$f(x^* + H) - x^* = AH + g(H).$$

If we consider the right hand side $AH + g(H)$ on its own we notice that it is zero if and only if $f(x) = x^*$ so that zero is a fixed point of $AH + g(H)$ if and only if x^* is an equilibrium of f. We also see that repeated iterations of the map $AH + g(H)$ go to zero if and only if repeated applications of the map f go to x^*, thus stability of one implies (and is implied by) the stability of the other. This result also holds for instability as well. Since it will prove more convenient we will work with H from here on.

Consider the case where the maximal modulus eigenvalue of the Jacobian matrix A for the system defined by the transition function f is less than one and consider the system associated with H. From our above lemma we know there is a symmetric matrix B such that $B - A^T BA$ is positive definite, we can make a Liapunov function $V(H) = H^T BH$. We are interested in the variation $\triangle V(H)$ we know that:

$$\triangle V(H(n)) = H^T(n+1)BH(n+1) - H^T(n)BH(n),$$

by letting $H = H(n)$ we see that this is equivalent to

$$(AH + g(H))^T B(AH + g(H)) - H^T BH,$$

using properties of the transponent operation and simple factoring we have that this is equivalent to

$$((AH)^T + g(H)^T)B(AH + g(H)) - H^T BH,$$

and by further manipulation

$$((AH)^T + g(H)^T)B(AH + g(H)) - H^T BH, = H^T(A^T BA - B)H + 2H^T A^T Bg(H) + g(h)^T Bg(h)$$

$$= H^T(A^T BA - B)H + 2H^T A^T Bg(H) + V(g(H))$$

we arrive at

$$-H^T(B - A^TBA)H + 2H^TA^TBg(H) + V(g(H)).$$

Since $B - A^TBA$ is positive definite "$H^T(B - A^TBA)H$" is simply the "$\|\cdot\|_{B-A^TBA}$" norm applied to H, and hence is equivalent to the standard $\|\cdot\|_1$ norm. Thus there exists α such that given any x, $\|x\|_{B-A^TBA} > \alpha \cdot \|x\|_1^2$ And since $g(H)$ goes to zero "supralinearly" (i.e. faster than H does) we can conclude that in a suitable region around the equilibrium $\bigtriangledown V(H) < 0$, and thus the equilibrium is stable. Now if $\rho(A) > 1$, where $rho(A)$ denotes the maximum modulus of the eigenvalues of A, the case is simpler. Assume we are working near enough to the equilibrium point x^* so that the $g(H)$ terms are negligible, so we can consider the system to be defined as $f(H) = AH + \epsilon$. We know that A has at least one eigenvalue λ such that $|\lambda| > 1$. Consider an open ball centered at the equilibrium of radius μ and the vector given by the normalized version of the eigenvector associated with λ (i.e $v_\lambda = \frac{v_\lambda}{\|\lambda\|}$) multiplied by the constant $c = \mu \cdot (\frac{1}{\lambda} + \frac{1}{2})$. Since λ is greater than one its obvious that this vector will be within a μ-neighborhood, but $f^n(cv_\lambda) = \lambda^{n-1}\mu(1 + \frac{\lambda}{2})v_\lambda > \mu v_\lambda$ for all n, so the orbit of v_λ will exit the μ-neighborhood, thus the equilibrium is unstable. There are ready examples for both stability and instability for the case of $\rho = 1$ available in [6] and thus no conclusion follows from our theorem in this case. □

3.3 Global Stability Properties for Higher Order Discrete Dynamical Systems: An Example

Examining the linearization of a system with continuous transition mapping can only tell us about local behavior of the system about a given equilibrium point. This is because as we move away from the equilibrium point the possible nonlinear terms of the function's Taylor series begin to have more and more influence on the qualitative dynamics of the system. Thus the system's linearization will not suffice to determine the system's global behavior

and stability properties for nonlinear cases. So far our only result that implies anything about the global properties of a system come from Liapunov theory. As it is generally quite difficult to come up with suitable Liapunov functions for a given system this too will not be overly helpful when investigations extend beyond the local case.

So what are we to do when we wish to understand global behavior of discrete dynamical systems? The answer (for now) is that we must consider other properties specific to the form of the systems in question and rely on restrictions implied by the phenomena which the system is modeling, such as the fact that populations must be non-negative, etc.

To begin we will work with a simple example from a one dimensional system and extend it to a more general, higher-order case. We follow [5] in examining a one dimensional difference equation of the form

$$x(n+1) = g(x(n)) \qquad x(0) = x_0 \tag{3.4}$$

under the assumption that all initial values x_0 are positive and that the mapping $g : R \to R$ is continuous and only give orbits with non-negative terms for any given positive initial value x_0. In addition we will assume that there is only a single equilibrium point,$\hat{x}$, for the system . Such a system will be referred to as a *population model*. We will use them as a base case for how to go about analyzing more complex cases. We will exploit certain monotonicity criteria which may be present in population models to help determine global properties. These results can be found in [5].

We now wish to explore a particular example of a higher order system to give the reader an idea of what is involved when trying to determine global stability properties. Assume we are working with a system which appears to be governed by a "delay-logistics equation" of the form

$$x_{n+1} = \frac{Ax_n}{1 + Bx_{n-1}} \qquad x(1) = x_1, \quad x(2) = x_2$$

with the parameters such that $A > 1$ and $B > 0$. A representative plot of the orbit of such an equation with parameter values $A = 10$, $B = 5$, $x(1) = 2$, and $x(2) = 1$ is given by Figure

3.1 below.

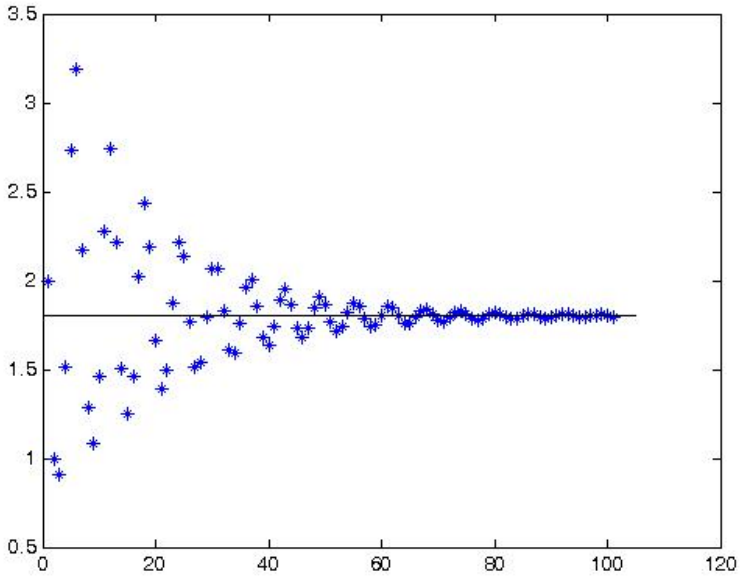

Figure 3.1: A=10, B=5, x(1)=2, x(2)=1

Investigation of many such numerical examples may help us to reverse engineer the source of the apparent stability properties. By tracing through repeated patterns in the numerical simulations we may be able to establish the underlying analytic properties which are causing the apparent stability behavior or we may generate a counterexample. We may then be able to generalize to other, less restricted cases. In the case above we notice a certain pattern of oscillations converging to an equilibrium point which repeats through numerous variations of the underlying parameters. We follow Kocic and Ladas [14] in presenting a generalization of the above to get global stability results for certain special systems with delays. We begin by introducing the second order nonlinear difference equation determined by the equation

$$x_{n+1} = x_n f(x_n, x_{n-1}) \qquad x(0) = x_0, \quad x(1) = x_1 \tag{3.5}$$

Note that we could just as easily treat the above equation as a two-dimensional system by letting $y_n = x_{n-1}$, however, for purposes of our analysis we will continue to work with the

the system in its above *delayed* form. In order to begin we need to introduce several new concepts, beginning with *oscillations* of an orbit of solutions.

Definition 8. *An orbit $\{x_n\}$ of a one dimensional system of difference equations is said to oscillate about and equilibrium $\hat{x}$ if the terms of $\{x_n - \hat{x}\}$ are neither eventually all positive, nor eventually all negative. It is said to strictly oscillate if it oscillates and for every $n > 0$, there exists $m, p > n$ such that $x_m - \hat{x}, x_p - \hat{x}$ are less than zero.[14]*

We can see that, if we require the definition above, it will be convenient not to write our second order difference equation in multivariable form.

When a system displays oscillatory behavior, we will like to focus on certain *"strings"* of the orbits. These "strings" are in essence finite subsequences of consecutive terms in the orbit. If an orbit oscillates about an equilibrium $\hat{x}$ a string of terms all of which are greater than or equal to $\hat{x}$ will be called a positive semi-cycle, while if a string of terms which contains elements less than $\hat{x}$ will be called a negative semi-cycle. Of course, we wish that our semi-cycles are maximal, if a positive (or negative) semi-cycle is properly contained in another positive(or negative) semi-cycle then we will only take in consideration the larger (and in the end, largest) semi-cycle in the chain. Now we will state and prove a result regarding the nonlinear second order difference equation above following [14]

Theorem 3. *Assume that f is restricted to positive real arguments and takes values in the positive real numbers, also assume that $f(u, v)$ is not increasing u and decreasing in v and that $u \cdot f(u, u)$ is increasing in u. Also assume that the system has a unique positive equilibrium $\hat{x}$, then $\hat{x}$ is globally asymptotically stable.*

Proof. To prove this we consider a positive initial condition for x_0 and x_1. To continue we need to establish several inequalities which follow from the assumed properties of f given above then, given $0 < a < \hat{x} < b$, we have that:

$$af(a, a) < \hat{x} \qquad bf(b, b) > \hat{x}$$

$$\hat{x}f(\frac{\hat{x}^2}{a},\frac{\hat{x}^2}{a}) > a \qquad \hat{x}f(\frac{\hat{x}^2}{b},\frac{\hat{x}^2}{b}) < b$$

$$\hat{x}f(\hat{x}f(a,a),\hat{x}f(a,a)) > a \quad \hat{x}f(\hat{x}f(b,b),\hat{x}f(b,b)) < b$$

The top row's inequalities follow from from the fact that $f(\hat{x},\hat{x})$ must be equal to one, and the assumed monotonicity of $uf(u,u)$. Since we assume that $a < \hat{x}$ it follows that $\frac{\hat{x}^2}{a} > \hat{x}$ and since for $b > \hat{x}$ it follows that $bf(b,b) > \hat{x}$ we have $\frac{\hat{x}^2}{a}f(\frac{\hat{x}^2}{a},\frac{\hat{x}^2}{a}) > \hat{x}$ and by simple algebra our middle left inequality is established. An analogous argument using the top-right inequality serves to establish the middle-right inequality. Also from simple algebraic manipulation of $af(a,a) < \hat{x}$ we can establish that $\hat{x}f(a,a) < \frac{\hat{x}^2}{a}$ and referring back to our assumption that $f(u,v)$ is nonincreasing in v, we establish the lower left (and by similar reasoning lower right) inequalities.

We also need to note some facts about semi-cycles which may occur in our system:

1. A positive semi-cycle cannot have two consecutive terms equal to $\hat{x}$ (since then all terms afterwards would be $\hat{x}$ and the semi-cycle would fail to be finite.)

2. Every semi-cycle after the first must have at least two terms. This follows because $f(u,v)$ is non-increasing in the second argument so that

$$x_{n+1} = x_n \cdot f(x_n, x_{n-1}) > x_n \cdot f(x_n, x_n)$$

and from the monotonicity of $uf(u,u)$ we know that $x_n \cdot f(x_n,x_n) > \hat{x}f(\hat{x},\hat{x}) > \hat{x}$

3. The maximum values (respectively minimum value) in a positive (negative) semi-cycle is either the first or second term of the semi-cycle. Moreover after the first term of the positive semi-cycle the remaining terms are non-increasing, while in a negative semi-cycle they are non-decreasing. This is because for two positive consecutive terms x_n, x_{n+1} we have

$$x_{n+2} = x_{n+1}f(x_{n+1},x_n) \leq x_{n+1}f(\hat{x},\hat{x}) = x_{n+1},$$

where the inequality results from the non-increasing and decreasing character of the first and second arguments respectively.

4. Except in the first semi-cycle (again due to our ability to choose the initial conditions arbitrarily) in any semi-cycle the maximum (or minimum) term cannot be equal to the last term of the semi-cycle. For the positive case we note that if the maximum is the last term say x_n then $x_{n-1} < x_n$ (since x_n is the maximum) and $x_{n+1} < \hat{x}$ (since it is the last term of a positive semi-cycled) but we know

$$x_{n+1} = x_n f(x_n, x_{n-1}) \geq x_n f(x_n, x_n),$$

from the fact that $f(u, v)$ is non-increasing in the second argument. While $x_n f(x_n, x_n) > \hat{x} f(\hat{x}, \hat{x})$ from our top right inequality and the fact that $f(\hat{x}, \hat{x})$ must be unity. But this means that $x_{n+1} > \hat{x}$ *and* $x_{n+1} < \hat{x}$ which is impossible, thus the last term of a positive semi-cycle cannot be the maximum term of that semi-cycle. Of course, the symmetric argument follows from for negative semi-cycles for our above inequalities.

5. For solution orbits for four or more semi-cycles, each successive maximum for the positive semi-cycles is decreasing, while each successive minimum for the negative semi-cycles is increasing.

   We can see this by considering a sequence of four semi-cycles with respective maxima/minima denoted M_1, M_2, M_3, M_4. Without loss of generality we assume that M_2 is the maximum of a positive semi-cycle. We must show that $M_4 < M_2$ while $M_1 > M_3$. From above we know M_2 must be the first or second term of its semi-cycle, if it is the first then $M_2 = x_l f(x_l, x_{l-1}))$ for x_l, x_{l-1} the final terms in the preceding negative semi-cycle and from the monotonicity properties of the arguments of f we can conclude that this is less than $\hat{x} f(M_1, M_1)$ while if it is the second value we get the same result with one additional step. Likewise for the negative semi-cycles we can obtain

$M_3 \geq \hat{x}f(M_2, M_2)$. So we have

$$M_4 < \hat{x}f(M_3, M_3) < \hat{x} < \hat{x}f(f(M_2, M_2), f(M_2, M_2)) < M_2,$$

while

$$M_3 > \hat{x}f(M_2, M_2) \geq \hat{x}f(\hat{x}f(M_1, M_1), \hat{x}f(M_1, M_1)) > M_1.$$

Trivially if the orbits of initial values for the system have a finite number of semi-cycles, then it must identically be $\hat{x}$ after some finite point, and thus the equilibrium $\hat{x}$ is globally asymptotically stable. So we will assume that there exists some initial condition x_0, x_1 such that the solution orbit generated will have an infinite number of semi-cycles. In that case, then from 5 above we know that $m = \liminf\{x_n\}$ and $M = \limsup\{x_n\}$ must both exist, and it must be that $0 < m \leq \hat{x} \leq M$, and that

$$M \leq \hat{x}f(\hat{x}f(M, M), \hat{x}f(M, M)),$$

and

$$m \geq \hat{x}f(\hat{x}f(m, m), \hat{x}f(m, m)).$$

Since in both cases these inequalities can be combined with our previously derived inequalities so that $M > \hat{x}$ implies $\hat{x}f(\hat{x}f(M, M), \hat{x}f(M, M)) < M$, thus it must be that $M = \hat{x}$. Likewise $m < \hat{x}$ implies that $\hat{x}f(\hat{x}f(m, m), \hat{x}f(m, m)) > m$ so it must be that $m = \hat{x}$. So we have proved asymptotic convergence, but we still have to prove stability. Assume that our initial conditions x_0, x_1 are within δ of our fixed point $\hat{x}$ where $0 < \delta < \hat{x}$, note that for x_0, x_1 it is the case that

$$\frac{-2\hat{x}\delta}{\hat{x} + \delta} < x_n - \hat{x} < \frac{2\hat{x}\delta}{\hat{x} + \delta}.$$

Thus for x_2 we have that $x_2 = x_1 f(x_1, x_0)$ and since f is monotonically decreasing in the second argument and non-increasing in the first argument we have $x_1 f(x_1, x_0) > (\hat{x} - \delta)f(\hat{x} +$

$\delta, \hat{x}+\delta) = \frac{\hat{x}-\delta}{\hat{x}+\delta}(\hat{x}+\delta)f(\hat{x}+\delta, \hat{x}+\delta)$ which from the fact that $\hat{x} < bf(b,b)$ for $b > x$ we can show is greater than $\frac{\hat{x}-\delta}{\hat{x}+\delta}\hat{x}$ which by simple algebra can be shown to be greater than $\hat{x} - \frac{2\hat{x}\delta}{\hat{x}+\delta}$. A symmetric chain of argument establishes that $x_2 < \hat{x} + \frac{2\hat{x}\delta}{\hat{x}+\delta}$.

We denote by C_0 the semi-cycle which contains x_1 and without loss of generality assume that C_0 is negative. Also assume that C_0 is not the last semi-cycle (if it is then it is then the point is trivial since the sequence must be then be constant after C_0) and denote the next semi-cycle C_1. We will then denote the minimum of C_0, x_{min} and the maximum of C_1, x_{max} from part 2 in our facts list above we know that for all n we have $x_{min} < x_n < x_{max}$. If x_0 is in C_0 then we know that $x_{min} = x_0$ or $x_{min} = x_1$. Clearly $x_{min} > \hat{x} - \delta > \hat{x} - \frac{2\hat{x}\delta}{\hat{x}+\delta}$, while we also have

$$x_{max} < \hat{x}f(x_{min}, x_{min}) < \hat{x}f(\hat{x}-\delta, \hat{x}-\delta) < \frac{\hat{x}}{\hat{x}-\delta}(\hat{x}-\delta)f(\hat{x}-\delta, \hat{x}-\delta),$$

which is less than

$$\frac{\hat{x}}{\hat{x}-\delta}\hat{x}f(\hat{x},\hat{x}) = \frac{\hat{x}^2}{\hat{x}-\delta} < \hat{x} + \frac{2\hat{x}\delta}{\hat{x}-\delta}.$$

So we can conclude that

$$\hat{x} - \frac{2\hat{x}\delta}{\hat{x}-\delta} < x_{min} \leq x_n \leq x_{max} < \hat{x} + \frac{2\hat{x}\delta}{\hat{x}-\delta}.$$

Now if x_0 is not in C_0 and x_2 must be in C_0 then it must be that $x_0 \geq \hat{x}$ (since otherwise it would be included in the leading negative semi-cycle) and $x_{min} = x_1$ or $x_{min} = x_2$ in the first case we can proceed just as above, in the second case we note that we have already shown that $x_2 > \hat{x} - \frac{2\hat{x}\delta}{\hat{x}-\delta}$ and $x_2 < \hat{x} - \frac{2\hat{x}\delta}{\hat{x}-\delta}$ so the conclusion again follows. If given an $\epsilon > 0$ we choose $\delta = \frac{\hat{x}\epsilon}{2\hat{x}+\epsilon}$ and thus $x_0, x_1 \in (\hat{x}-\delta, \hat{x}_\delta)$ implies that $|x_n - \hat{x}| < \epsilon$ for all n. Thus $\hat{x}$ is stable and the proof is complete. □

The tediousness of the above proof gives evidence to the fact that "brute force" analysis of non-linear discrete dynamical systems thus far yields results regarding global stability of

equilibria only by exploiting specific analytic properties of the governing equations. Our only universally general result regarding global stability in these systems comes from Liapunov theory, and there we face the nontrivial problem of determining a suitable Liapunov function for a given system. Usually in mathematics "ad hoc" methods like those in the example breed a justified sense of dissatisfaction and we search for more systematic approaches. One such approach to the problem is offered by considering systems as they relate to *semiconjugates* of maps of the real line. For further results in this area readers are encouraged to refer to [25].

Chapter 4

Bifurcations

If we restricted our discussion to linear systems the only phenomena of interest would be the (comparatively) placid behavior of the systems equilibrium, which would be relegated to the origin. However, in the nonlinear case more sophisticated phenomena manifest. The focal point of interest in dynamical systems in the past several decades has been the phenomena of *bifurcation* that we lightly touched on in the introductory sections when discussing the logistics map. In chapter 3 we were able to analyze the behavior of equilibria when the maximum modulus of the Jacobian's eigenvalues was not unity. In these cases the equilibrium in question is said to be *hyperbolic*. In cases where the equilibrium is not hyperbolic the linearization may fail to reveal the behavior of the system on certain subspaces of the state-space. These subspaces are associated with the eigenvectors of the Jacobian matrix with eigenvalue $\lambda = 1$.

The theory resulting from the analysis of these subspaces is called *Center Manifold Theory*, and the specific case of Center Manifold Theory for the two dimensions will be expounded in the next subsection. Our presentation will follow that given in [6].

4.1 Center Manifold Theory

We will consider a discrete dynamical system with transition function defined from the state-space $R^2 \to R^2$, with an associated parameter space a subset of R^n. Our analysis will concern what qualitative changes happen to the system's equilibria as we vary the input parameters. We will denote this system

$$x(n+1) = F(\mu, x(n)) \qquad x(0) = x_0, \tag{4.1}$$

where the μ is vector of parameters. For the purposes of our analysis we will also assume that $F(\mu, x)$ has continuous derivatives at least up to the third degree. For a given value of μ say μ^* the point x^* is said to be an equilibrium of the system (4.1) if

$$F(\mu^*, x^*) = x^*.$$

When the Jacobian matrix, A, of this system exists and the maximum modulus of the eigenvalues is not equal to one then we can refer to the previous section's theorem to sufficiently analyze the system about its equilibrium points. To handle the case where $\rho(A) = 1$ we first fix μ and by a suitable change of coordinates, translate the equilibrium to the origin and use as basis vectors the eigenvectors of the Jacobian matrix with x being the basis vector associated with $\lambda = 1$. Then since we are in the plane we consider the system (4.1) to be represented as, given a point (x,y)

$$\begin{aligned} x &\mapsto x + f(x, y) \\ y &\mapsto by + g(x, y), \end{aligned} \tag{4.2}$$

which in our discrete context will be

$$x(n+1) = x(n) + f(x(n), y(n))$$
$$y(n+1) = by(n) + g(x(n), y(n).$$

We can now state the Center Manifold Theorem following [6]:

Theorem 4 (Center Manifold Theorem)**.** *Assume that we are given a system of the form (4.2) Then there is a continuously differentiable (to at least the third degree) center manifold M which can be represented locally about the equilibrium (which has been translated to the origin via change of basis) as*

$$M = \{(x, y) : y = h(x), |x| < \delta, h(0) = 0, Dh(0) = 0\}$$

for sufficiently small δ. The behavior of the system on M is locally equivalent to the mapping

$$x \mapsto x + f(x, h(x))$$

Thus if

$$x(n+1) = x(n) + f(x(n), h(x(n)))$$

is stable at the origin so is the overall system. Likewise, asymptotic stablity or instablity follow from the asymptotic stability or instability of the system on the center manifold. We have thus been able to reduce analysis of our two-dimensional systems to analysis of its behavior on a one-dimensional manifold. In general this manifold will be approximated by its Taylor series expansion and we will be able to derive results from that.

4.2 Bifurcation Results in Two Dimensions

When dealing with a two-dimensional discrete dynamical system given by the equation

$$x(n+1) = F(x(n), \mu) \tag{4.3}$$

about a fixed point x_0 where x is in R^2, μ is a real-valued parameter constant, and f is sufficiently smooth, we have several guiding results. The linearization results from chapter 3 follow if the associated Jacobian matrix $J = DF(x_0)$ has no eigenvalues equal to 1. If J has an eigenvalue exactly equal to 1 we must consider the center manifold defined as $h(x, \mu)$ and note the following three cases

1. The system has a saddle-node bifurcation if $\frac{\partial h}{\partial \mu}(x_0) \neq 0$ and $\frac{\partial^2 h}{\partial \mu^2}(x_0) \neq 0$
2. The system has a pitchfork bifurcation at the equilibria if $\frac{\partial h}{\partial \mu}(x_0) = 0$ and $\frac{\partial^2 h}{\partial \mu^2}(x_0) = 0$
3. The system has a transcritical bifurcation if $\frac{\partial h}{\partial \mu}(x_0) = 0$ and $\frac{\partial^2 h}{\partial \mu^2}(x_0) \neq 0$.

If J has an eigenvalue equal to -1 then the system will have a period-doubling bifurcation at the equilibrium. If there is a pair of complex eigenvalues with modulus 1, then the system will have a "Neimark-Sacker" bifurcation which corresponds to the Hopf-bifurcation from ordinary differential equations and is sometimes referred to as a "discrete Hopf" bifurcation. Now would be a good time to go over just what these types of bifurcation entail in a two dimensional system with some elementary examples. Note that with our two dimensional system with state-space R^2 the system's behavior about a fixed point x_0 will depend on its behavior on the one-dimensional center manifold if the eigenvalues are real, thus it suffices to investigate the bifurcation behavior in one dimension unless a pair of complex conjugate eigenvalues exist for the linearized system.

4.2.1 Saddle-node,Transcritical, and Pitchfork Bifurcation

The first type of bifurcation which we will discuss is a saddle-node bifurcation. As stated above we are investigating the behavior of the system on a one dimensional center subspace so that we are interested in the behavior of

$$x(n+1) = x(n) + f(\mu, x(n), h(x(n))) \tag{4.4}$$

about an equilibrium point x_0 where h is a function defining the center subspace. For notational convenience we will use the fact that h is a function of x to suppress it and consider f with regard to its explicit x dependency, i.e. $f(\mu, x(n), h(x(n)) = f(\mu, x)$. We are now ready to state the conditions for saddle-node bifurcation to occur.

Definition 9. *Let $f(\mu, x)$ as above be sufficiently smooth so as to have two continuous derivatives with regard to both x and μ then if:*

1. $f(x_0, \mu_0) = x_0$
2. $\frac{\partial f}{\partial x} = 1$ *at* (x_0, μ_0)
3. $\frac{\partial^2 f}{\partial x^2} \neq 0$ *at* (x_0, μ_0)
4. $\frac{\partial f}{\partial \mu} \neq 0$ *at* (x_0, μ_0)

then we say that a saddle-node bifurcation has occurred at (x_0, μ_0).

When a saddle-node bifurcation occurs in two dimensions a single equilibrium point of the system bifurcates into two distinct equilibria points with opposite stability properties, thus by analogy to "saddles" from multivariate calculus we get the name "saddle-node".

We now present a simple example of a system exhibiting a saddle-node bifurcation. Consider a system such that on the center manifold dynamics are determined by the equation

$$f(x) = \mu + x - x^2$$

at the point (0,0). We certainly have $f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) = 1$. While $\frac{\partial^2 f}{\partial x^2} = -2 \neq 0$ and $\frac{\partial f}{\partial \mu} = 1 \neq 0$. So a saddle-node bifurcation will occur. In this instance as μ passes from negative to positive values the system goes from having no equilibria to a single unstable fixed point at the origin, to having two equilibria at $x = \pm\sqrt{\mu}$ with the negative branch unstable and the positive branch stable.

The saddle-node bifurcation requires that $\frac{\partial f}{\partial \mu} \neq 0$ at the equilibrium of interest. If this is not the case, i.e. if $\frac{\partial f}{\partial \mu} = 0$ but otherwise the same condition hold then either a *transcritical* or *pitchfork* bifurcation may occur. If $\frac{\partial^2 f}{\partial x^2} \neq 0$ then we have a transcritical bifurcation. Under a transcritical bifurcation two distinct equilibria "exchange" stability characteristics. Thus if x_0 were a stable equilibrium and x_1 an unstable equilibrium for $\mu < \mu_0$ where μ_0 is a bifurcation value, then for $\mu > \mu_0$, x_1 would be stable while x_0 would lose its stability. If $\frac{\partial^2 f}{\partial x^2} = 0$ then we have a pitchfork bifurcation. In this case a stable (re: unstable) equilibrium changes stability and gives off two new branches of equilibria which take on its previous stability characteristics.

4.2.2 Period Doubling Bifurcations

We are now going to investigate bifurcation behavior near an equilibrium when the center manifold of the system at the fixed point is associated with an eigenvalue of -1. Here we are interested in the behavior determined by

$$x(n+1) = f(\mu, x(n), h(x(n)))$$

written in terms of its x and μ dependency.

Definition 10. *Assume that we have*

1. $f(x_0, \mu) = x_0, \forall \mu \in (\mu_0 - \epsilon, \mu_0 + \epsilon)$ *for some* μ_0 *and* $\epsilon > 0$
2. $\frac{\partial f}{\partial x}(x_0, \mu_0) = -1$, *and*
3. $\frac{\partial^2 f^2}{\partial \mu \partial x}(x_0, \mu_0) \neq 0$

Then there is an interval I *about* x_0 *and a function* $m(x) : I \to R$ *from the interval into the parameter space such that* $f(x, m(x)) \neq x$ *but* $f^2(x, m(x)) = x$. *Here we say that a period doubling bifurcation has occurred.*

An example of such a bifurcation is given by the equation

$$f(x, \mu) = -\mu x + x^3$$

Notice that $(0, 1)$ is an equilibrium for the given system and that this is the case regardless of μ, also with $\mu = 1$ the derivative with respect to x at (0,1) is -1, thus our first two conditions are easily verified. For the last condition we need to find $f(f(x, \mu), \mu)$, which is simply

$$f^2(x, \mu) = -\mu(-\mu x + x^3) + (-\mu x + x^3)^3$$

so we can see that

$$\frac{\partial^2 f^2}{\partial \mu \partial x}(0, 1) = 2 \neq 0$$

so we conclude that a period doubling bifurcation has occurred and a new orbit of period-two has emerged.

4.2.3 Neimark-Sacker Bifurcation

When the Jacobian matrix of our linearized system has two complex conjugate eigenvalues of modulus one a *Neimark-Sacker bifurcation*, sometimes called a discrete Hopf bifurcation, will occur. The dynamics of this bifurcation are similar to a Hopf bifurcation, in that an equilibrium alters stability properties and a closed, invariant curve emerges. The criteria for a Neimark-Sacker bifurcation are now given:

Definition 11. *Let $f(x, \mu)$ be sufficiently smooth so as to have continuous derivatives to at least the fifth order. And let the following conditions hold:*

1. *$f(0, \mu) = 0$ so that the origin is a fixed point of the system*

2. *The Jacobian matrix $Df(0, \mu)$ has a pair of complex conjugate eigenvalues depending on μ of the of the form $\lambda(\mu) = r(\mu)e^{i\theta(\mu)}$ where $r(0) = 1$, $\frac{dr}{d\mu} \neq 0$ and $\theta(0) = \theta_0$ So that*

the modulus at $\mu = 0$ is 1

3. $e^{ik\theta_0} \neq 0$ *for* $k < 5$

Then for small μ, the system defined by f generates a closed, invariant curve around the origin if $\frac{\mu}{f(\mu)} > 0$. If $f(0,\mu) > 0$ then it is attracting, while if $f(0,\mu) < 0$ it repels. We say that a Niemark-Sacker bifurcation has occurred.

For example consider the two-dimensional discrete dynamical given in polar coordinates by

$$r(n+1) = (1+\mu)r(n) + [r(n)]^3$$
$$\theta(n+1) = \theta(n) + \beta.$$

As μ goes from negative to positive values the origin equilibrium loses its stability and an attractor appears–an invariant circle of radius $\sqrt{\mu}$.

4.2.4 Contact Bifurcation of Basins of Attraction

For the sake of comparison we begin this section by considering some geometric properties of continuous dynamical systems[1]. Consider the system defined by the equations

$$\frac{dx}{dt} = f(x,t) \tag{4.5}$$

where t is a continuous time parameter, x is vector in the state space R^n and $f(x,t)$ is a continuous function from $R^{n+1} \to R^n$, for any given initial state x_0, the solution trajectory, $\Gamma(x_0)$ is defined to be the set of all points x in the state space such that

$$x(t) = x_0 + \int_0^t f(x,s)ds$$

for some $t \in [0,\infty)$. We have already noted that it is impossible for a trajectory in a continuous system to intersect an equilibrium point unless it begins on that equilibrium

[1]Material in this section is adapted from [1]

point. It should also be noted that for distinct starting point, x_0 and y_0 the trajectories $\Gamma(x_0)$ and $\Gamma(y_0)$ cannot intersect in finite time. If S is a given set in the state space and as $t \to \infty$ the solution of (4.5), ϕ, approaches the set S in the given metric, S is said to be the "ω-limit set" of ϕ. If we take S the set of $x_0 \in R^n$ such that S is the ω-limit set of $\phi(x_0)$ is called the *basin of attraction* for S. In the continuous case the fact that solution trajectories can not overlap and that initial conditions which are not equilibria can never actually reach equilibria in finite time prevents basins of attraction from varying geometrically too much from the topological structure of the attractor set. However, as we have shown earlier in the discrete case it is possible for the solution trajectories (i.e. the orbits) to reach equilibrium points in finite time, likewise the orbits generated by different initial conditions in discrete dynamical systems have a greater degree of freedom in regards to "crossing paths" with the orbits of other initial conditions. This allows for basins of attraction with much more complex geometric and topological properties than the attractor set with which they are associated. Much like the equilibria themselves, these basins of attraction can undergo significant qualitative behavioral changes as parameters are varied, meaning that they too can be the subject of bifurcation analysis. To begin our analysis we will introduce the intuitive notion of a "non-invertible" map $T : R^n \to R^n$, where R^n is the underlying state space for our system. For a given $\hat{x}$ in R^n we define the set $T^{-1}(\hat{x})$ as the set of all $x \in R^n$ such that $T(x) = \hat{x}$, that is the *preimage* of $\hat{x}$ under T. A map is invertible if it is a function and the preimage of each singleton set in the range of T is itself a singleton set. Thus a non-invertible map is a function for which the preimage of a singleton in the range is a non-singleton (and non-empty) set. In layman's terms multiple elements in the domain map to the same element in the range.

We want to investigate the behavior of the discrete dynamical system defined by:

$$x_{n+1} = T(x(n)) \qquad x(0) = x_0 \tag{4.6}$$

where T may be a non-invertible map. We take the set S to be an *attractor* for the above system, i.e. a closed invariant locally asymptotically stable set under our usual definitions and let $\mathbf{B}(S)$ the basin of attraction for S. If we let U be an open neighborhood of S such that $T^n(x) \to S$ as $n \to \infty$ for all x in U then

$$\mathbf{B}(S) = \cup_{n=0}^{\infty} T^{-n}(U)$$

We define the rank 1 preimage under T of a point $\bar{x}$ in our state space X as the set of all x_i in X such that $T(x_i) = \bar{x}$, the rank 2 preimage as the set of all x_i such that $T^2(x_i) = \bar{x}$, and so on. We can partition the state space, X, into distinct regions $Z_0, Z_1, ...Z_k$ denoting that the points in Z_k each have k rank one preimages, that is each point x in Z_k is mapped to by one iteration of T from k distinct points in X.

We now wish to generalize a basic notion from first year calculus, that of a local maximum/minimum and a critical point to help us understand the bifurcations that can occur with regard to basins of attraction. If we consider the one dimensional map defined by $f(x) = -(x-1)^2+1$ on the interval $[0, 2]$ so that the range of f is contained in $[0, 2]$. Clearly $f(x)$ is not invertible on the given domain, since for every value in $[0, 1)$ there are two distinct points in $[0, 2]$ which map to that value under f. The point of interest in the co-domain is $\{1\}$ mapped to by 1 in the domain. In elementary calculus the point $\{1\}$ in the domain is called a "critical point" while $\{1\}$ in the codomain is the local max of f. For generalization purposes, we are interested in what happens as we vary from zero to one the codomain. Points in $[0, 1)$ have two preimages which, as we approach 1, collapse into a single preimage, and for values in $(1, 2]$ there are no preimages . In our established notation $[0, 1]$ could be partitioned into $Z_2 = [0, 1)$ and $Z_0 = (1, 2]$. Thus the local maximum is the boundary of the Z_k sets, and the point which the function takes to the local max is called a *critical point*. We are now ready for our generalization:

Definition 12. *The critical set, CS, is the set of all points which, under a given transfor-*

mation T, have at least two rank-1 preimages which come together, located on CS^{-1}, the "set of merging preimages" so that $T(CS^{-1}) = CS$.

A critical set separates a subset of the systems state space into say a Z_{k+2} region and a Z_k region. For points x in Z_{k+2}, $k+2$ preimages exist, which we will denote by $\{T_1^{-1}, T_2^{-1}, ..., T_{k+1}^{-1}, T_{k+2}^{-1}\}$, say T_1^{-1}, T_2^{-1} are the two preimages which merge, then the respective ranges of T_1^{-1}, T_2^{-1} are in different regions of Z_{k+2} separated by the set CS^{-1}, and on CS^{-1} these preimages come together on their image CS and they disappear as we cross CS into Z_K. Analogous to the one dimensional example above, where "critical points" in the domain of the function map to local maximums or minimums in the range and $\frac{df}{dx} = 0$ at these points, if f is a smooth map then the preimage of the critical sets CS^{-1} will have a determinant which evaluates to zero for the Jacobian of f. Likewise points of discontinuity or non-differentiable kinks may also belong to CS^{-1}.

We will present a one-dimensional example of how these critical sets can cause interesting behavior and "contact" bifurcations in nonlinear systems basins of attraction. Consider the previously encountered logistics map from above defined by

$$x(n+1) = \mu x(n)(1 - x(n)) \tag{4.7}$$

We solve for the non-zero equilibrium

$$x(n) = \mu x(n)(1 - x(n))$$

$$\frac{1}{\mu} = 1 - x(n)$$

$$\rightarrow x(n) = \frac{\mu - 1}{\mu}$$

The critical set of the map is the local max which is located at $\{\frac{\mu}{4}\} = CS$ which divides the co-domain of the map into two regions $Z_2 = (-\infty, \frac{\mu}{4})$ and $Z_0 = (\frac{\mu}{4}, \infty)$, while $CS^{-1} = \{\frac{1}{2}\}$. If μ is less than 4, initial conditions within the interval $[0, 1]$ will generate bounded

orbits with attractors varying from our above equilibrium to chaotic periodic orbits at μ. The basin of attraction for these attractors however remains the interval $[0,1]$. How will the basin of attraction change as the parameter μ varies above 4? Letting $I = [0,1]$ and $f : R \to R$ be our mapping then we are interested in the preimage of I under f, $f^{-1}(I)$. The x_0 in I which will exit I after one iteration are those in the interval $\left(\frac{1-\sqrt{1-\frac{4}{\mu}}}{2}, \frac{1+\sqrt{1-\frac{4}{\mu}}}{2}\right)$, which will be empty of real solutions when $\frac{4}{\mu} > 1$ i.e. $\mu < 4$. Thus the interval is stable under the logistics map when μ is less than 4. As μ varies past 4, a portion of the image of I under l, which we'll call $f^{-1}(I^c) \cap I$ now leaves I after one iteration of the mapping. Since the Z_2 region of the co-domain follows $\frac{\mu}{4}$ up this portion is in Z_2 and so has two preimages, each of which must be contained in I, and must be an interval (since f is continuous with respect to x), each of these has two preimage intervals in I and so on. If we examine $\cup_{k=0}^{\infty} f^{-k}(I^c) \cap I$ we can see using geometric analysis that this is the complement of a Cantor set, thus points in I whose orbits remain in I, i.e. the basin of attraction for our chaotic attractor, will themselves form a Cantor set.

Part II

Applications

Chapter 5

Applications to Biology

5.1 Some Elementary Examples

The application of discrete dynamical systems to biology predates both subjects. In one of the earliest entries into the canon of modern western mathematics Leonardo Pisano (Fibonacci), along with introducing Arabic numerals to the Italians, posed the question *"How Many Pairs of Rabbits Are Created by One Pair in One Year"*. In his own words:

> *"A certain man had one pair of rabbits together in a certain enclosed space, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also. Because the above written pair in the first month bore, you will double it; there will be two pairs in one month. One of these, namely the first, bears in the second month, and thus there are in the second month 3 pairs; of these in one month 2 are pregnant, and in the third month 2 pairs of rabbits are born, and thus there are 5 pairs in the month..."[8]*

In essence, each mature pair of rabbits produces a new pair of rabbits each month, and it takes a month for a rabbit to attain maturity and then be able reproduce itself. The sequence generated by counting the number of pairs of rabbits is $\{1, 2, 3, 5, 8, 13, 21, 34, ...\}$, and we

can see that the rule for generating the next term is to add the previous two terms, *"we add the first number to the second, namely the 1 to the 2, and the second to the third, and the third to the fourth...and thus you can in order find it for an unending number of months."* In our notation the governing transition equation for the recurrence relation is given by

$$x(n+1) = x(n) + x(n-1), \quad n > 1, \qquad x(0) = 0, x(1) = 1.$$

Let $y(n) = x(n-1)$ and we get the system of equations:

$$\begin{aligned} x(n+1) &= x(n) + y(n) \\ y(n+1) &= x(n), \end{aligned}$$

which we can recognize as a simple linear system:

$$\begin{pmatrix} x(n+1) \\ y(n+1) \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x(n) \\ y(n) \end{pmatrix}$$

which has the characteristic equation $\lambda^2 - \lambda - 1$ which implies that the eigenvalues are $\lambda = \frac{1 \pm \sqrt{5}}{2}$ with associated eigenvectors $v_{\lambda_1} = \left(\frac{1+\sqrt{5}}{2}, 1\right)$ and $v_{\lambda_2} = \left(\frac{1-\sqrt{5}}{2}, 1\right)$. We are given the initial condition (x(0),y(0))=(1,0), so we can find the closed form solution by projecting the initial condition onto the eigenvectors and taking the linearized matrix action on the vector in that form. i.e we will use the fact that $(1,0) = \frac{1}{\sqrt{5}} \cdot v_{\lambda_1} - \frac{1}{\sqrt{5}} \cdot v_{\lambda_2}$. So letting

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and recalling from earlier that the solution to a linear constant coefficient discrete dynamical system is given by

$$x(n) = A^n \cdot x_0,$$

which for our system can be written as

$$A^n\left(\frac{v_{\lambda_1}}{\sqrt{5}}-\frac{v_{\lambda_2}}{\sqrt{5}}\right)=\lambda_1^n\cdot\frac{v_{\lambda_1}}{\sqrt{5}}-\lambda_2^n\cdot\frac{v_{\lambda_2}}{\sqrt{5}}.$$

Since we are only explicitly interested in the resulting vectors first coordinate we note that the n^{th} Fibonacci number is given by:

$$x(n)=\frac{1}{\sqrt{5}}\cdot\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Of course this is an elementary example of a linear discrete population model and only models the endogenous growth of an idealized single species population. More interesting, nonlinear phenomena arise when we look at the more complex interactions between competing species. In the discrete time context we can consider the famous Lotka-Volterra equations given by:

$$\begin{aligned}x(n+1)&=a\cdot x(n)+b\cdot x(n)y(n)\\y(n+1)&=c\cdot y(n)+d\cdot x(n)y(n)\end{aligned}\tag{5.1}$$

Here the x and y state variables represent the population levels of two competing species, while the parameters represent the effect of endogenous growth (for a and c) and exogenous interaction between the species (for b and d). In essence the a coefficient can be seen as the natural growth rate for the species represented by the x variable, while the b coefficient is the penalty or boon (for negative or positive values respectively) incurred from interaction with the "y" species. The c and d coefficients play the same respective roles for the y species. Thus the system as a whole considers the growth in population to be proportional to current population and proportional to the number of interactions occurring between species, which all things being equal will be proportional to their product $(x\cdot y)$. Two equilibrium points for the system are readily seen, the origin and $(\frac{1-c}{d},\frac{1-a}{b})$. To analyze the system we can look

at the Jacobian matrix for (5.1)

$$J = \begin{bmatrix} a + b \cdot y(n) & b \cdot x(n) \\ d \cdot y(n) & c + d \cdot x(n) \end{bmatrix}.$$

At the origin the eigenvalues of this matrix will be a and c, thus if the modulus of both a and c is less than 1 (so neither species is self sustaining or self-propagating) then the system will decay to extinction. If either of the coefficients is greater than one, at least that species will locally flourish and its population will expand around the origin, since any depressing effect from interactions with the other species will be dampened significantly by the low population level, thus the origin will be unstable. If the largest of a and c is equal to 1 then the stability of the origin will depend on the center manifold that emerges and more interesting phenomena may occur. We now wish to present two examples from the literature that extend this basic concept in biological population modeling.

5.2 Example: A Two Dimensional System Modeling Cooperation

In this section we wish to present a less elementary example involving "rational" nonlinearities used to model biological systems where two populations are cooperating. We follow the example of Kulenovic and Nurkanovic in [15] and examine the system represented by the equations:

$$x(n+1) = Ax(n) \cdot \frac{y(n)}{1 + y(n)} \qquad y(n+1) = By(n) \cdot \frac{x(n)}{1 + x(n)} \tag{5.2}$$

where A and B are positive real constants.

We note that each of the state variables' transition functions are monotonically increasing in the other variable, thus we can say that this models cooperative behavior; each specie's population increase in turn helps to increase the other specie's population. Also we should

note that the nonlinearity in the system's transition functions are rational polynomials,which are a relatively well understood subset of nonlinear expressions [14].

By inspection we can see that the equilibria of the system occur at the points (0,0) and simple algebra shows that $\bar{x} = (\frac{1}{B-1}, \frac{1}{A-1})$ will be an equilibrium when A or B do not equal 1. Given our biological restrictions (i.e. no negative populations) we can thus conclude that the origin is always an equilibrium for the system, while there will be another admissible equilibrium at $\bar{x}$ when A and B are both greater than 1. Assuming that negative population levels are impossible, and thus restricting ourselves to the the first quadrant, simple linearization analysis shows that the origin is a locally stable equilibrium. We wish to establish that the equilibrium at $\bar{x}$ is locally unstable. We proceed by analysis of different cases for values of the equilibrium $\bar{x}$ as follows:

1. $A < 1, B < 1$. In this case we can see that, under our restrictions, we'll have that

$$x(n+1) = Ax(n)\frac{y(n)}{1+y(n)} \leq Ax(n)$$

   regardless of our initial value for $x(0)$. This together with the equivalent inequality for the y values shows that $A < 1$ and $B < 1$ implies that all solution trajectories will approach the origin. Thus the equilibrium $\bar{x}$ is unstable while the origin is globally asymptotically stable.

2. $A = B = 1$. In this case $\bar{x}$ is nonexistent, so the only equilibrium for the system is the origin. Also, since in such a case

$$x(n+1) = x(n)\frac{y(n)}{1+y(n)} \leq x(n),$$

   and

$$y(n+1) = y(n)\frac{x(n)}{1+x(n)} \leq y(n),$$

   we have that any given orbit is monotonically decreasing in both components, thus the

origin is globally asymptotically stable.

3. $A = 1$, $B < 1$ (or $B = 1$, $A < 1$). By symmetry apparent in the system it is only necessary for us to pursue one of these two possible parameter combinations. As in our second case above the origin is the only possible equilibrium of the system, and we have that $x(n+1) \leq x(n)$, while $y(n+1) \leq y(n)$, so again we see that the origin is asymptotically stable.

4. $A > 1, B > 1$. In this case we have that both the origin and $\bar{x}$ are equilibria for the system. Here the analysis become more interesting and less trivial than above. Locally the origin is again asymptotically stable, while $\bar{x}$ is a saddle. To show this, we define the mapping $T : (x, y) \mapsto \left(Ax\frac{y}{1+y}, By\frac{x}{1+x}\right)$ and introduce a partial order on the first quadrant by $(x_1, y_1) \leq (x_2, y_2) \iff x_1 \leq x_2$ and $y_1 \leq y_2$, with '$<$' meaning $\leq$ but no equality and strong inequality if both $x_1 < x_2$ and $y_1 < y_2$. We now can present some useful facts:

   First we note that T defined above is a monotone nondecreasing map with respect to our ordering of the first quadrant [16]. Thus for a given initial value (x_0, y_0), we have that

   (a) $(x_0, y_0) \leq T(x_0, y_0) = (x_1, y_1)$ then $(x_n, y_n) \leq (x_{n+1}, y_{n+1}), \forall n \geq 0$, and

   (b) $(x_0, y_0) \geq T(x_0, y_0) = (x_1, y_1)$ $\forall n \geq 0$ implies that $(x_n, y_n) \geq (x_{n+1}, y_{n+1})$.

   Each of these vector inequality relationships are equivalent to the individual variable sequences $\{x_n\}$ and $\{y_n\}$ being non-increasing/nondecreasing respectively. Thus we have:

$$(x_0, y_0) \leq (x_1, y_1) = \left(Ax_0\frac{y_0}{1+y_0}, By_0\frac{x_0}{1+x_0}\right)$$

   is true if and only if $(x_0, y_0) \geq (\frac{1}{B-1}, \frac{1}{A-1})$ component-wise. So that if $x_0 \geq \frac{1}{B-1}$ and $y_0 \geq \frac{1}{A-1}$ each of the respective x_n and y_n sequences will be nondecreasing, while if these are taken as strict inequalities then they will be increasing and we will get

divergence to infinity in each component.

Likewise, we have that

$$(x_0, y_0) \geq (x_1, y_1) = \left(Ax_0 \frac{y_0}{1+y_0}, By_0 \frac{x_0}{1+x_0} \right)$$

if and only if $(x_0, y_0) \leq (\frac{1}{B-1}, \frac{1}{A-1})$ component wise. In which case they will either remain at the $\bar{x}$ equilibrium (if equality holds) or decrease to the origin.

Under the current conditions we will have a stable manifold W^s associated with $\bar{x}$. We consider what happens to the "quadrants" generated by $\bar{x}$, that is, considering the usual I, II, III, IV, about the origin, only with $\bar{x}$ being the center of our plane (and restricting our area of inquiry to the first quadrant in the usual terms). For example, our relativized quadrant IV will be $\left\{(x, y) | x > \frac{1}{B-1},\ 0 < y < \frac{1}{A-1}\right\}$ and our relativized quadrant I would be $\left\{(x, y) | x > \frac{1}{B-1},\ y > \frac{1}{A-1}\right\}$. In this case the first and third quadrants thus relativized to $\bar{x}$ are invariant under our T mapping. We can see from some simple investigation that for initial conditions in our (relativized to $\bar{x}$) third quadrant have trajectories converging to the origin, while those in the first quadrant diverge to infinity, so both of these quadrants form invariant subsets under the iteration mapping. Thus the stable manifold W^s must be a subset of quadrant II or quadrant IV (or both).

Now assume that we are given an initial condition (x_0, y_0) in our relativized quadrant IV, so that $x_0 > \frac{1}{B-1}$ and $0 < y_0 < \frac{1}{A-1}$. Then if $y_1$is in our relativized quadrant I, it must be that $y_1 > \frac{1}{A-1}$. Now since

$$y_1 = By_0 \frac{x_0}{1+x_0} > \frac{1}{A-1},$$

we have that

$$By_0x_0(A-1) > 1 + x_0 > 1 + \frac{1}{B-1} = \frac{B}{B-1},$$

so

$$By_0x_0(A-1) > \frac{B}{1-B},$$

which implies that

$$Ay_0x_0 > y_0x_0 + \frac{1}{B-1} > (1+y_0)\frac{1}{B-1}$$

which is true if and only if

$$Ax_0\frac{y_0}{1+y_0} > \frac{1}{B-1}$$

which says that $x_1 > \frac{1}{B-1}$. So $T(x_0, y_0)$ is in the relativized quadrant I.

Likewise, if (x_0, y_0) is in quadrant IV, if $x_1 < \frac{1}{B-1}$ then (x_1, y_1) is in quadrant 3. If our initial condition is in quadrant II, then we have similar results, i.e. if $y_1 < \frac{1}{A-1}$ then (x_1, y_1) is in quadrant III, and if $x_1 > \frac{1}{B-1}$ then (x_1, y_1) is in quadrant I.

We now can see that if A and B are both greater than 1 the stable manifold of the equilibrium $\left(\frac{1}{A-1}, \frac{1}{B-1}\right)$,which we will denote W^s, partitions the (non-relativized) quadrant I into two invariant regions, is contained in the union of the (relativized) quadrant I and quadrant IV. Moreover orbits which start in the region which is "below"(i.e. the set of initial conditions x_0 such that $x_0 < x$ and $y_0 \leq y$ or $y_0 < y$ and $x_0 \leq x$ for some (x,y) in W^s is the basin of attraction for the origin, while the points in the region above (in as the same since but with the inequalities switched) will generate orbits diverging to infinity.

5. $A = 1, B > 1$. The final case of interest occurs when $A = 1, B > 1$ (or analogously $A > 1, B = 1$) in this case the origin is the only equilibrium, however, the global dynamics involved are more complex than may appear initially. We can see that given any admissible initial condition (x_0, y_0) the sequence of the x_n terms is given by

$$x_{n+1} = x_n\frac{y_n}{1+y_n} < x_n$$

hence the x coordinate components of an orbit should converge to zero regardless of starting point. However, where the x component begins has an effect on the future behavior of the y_n sequence. If $x_n \leq \frac{1}{B-1}$ then we have

$$y_{n+1} = By_n \frac{x_n}{1+x_n} \leq y_n.$$

To see this, let

$$C = \frac{Bx_n}{1+x_n}.$$

Then we have that

$$C + Cx_n = Bx_n \rightarrow C = x_n(B-C) < \frac{B-C}{B-1},$$

so it must be that

$$C(B-1) = CB - C < B - C,$$

thus $CB < B$ and we conclude $C < 1$ regardless of n, thus the y_n sequence thus must go to zero and orbits in this given region must converge to the origin.

The more interesting case arise for initial conditions in which the x component is greater then $\frac{1}{B-1}$. Analytically these dynamics are not certain, though Kulenovic and Nurkanovic suggest the following conjecture is true:

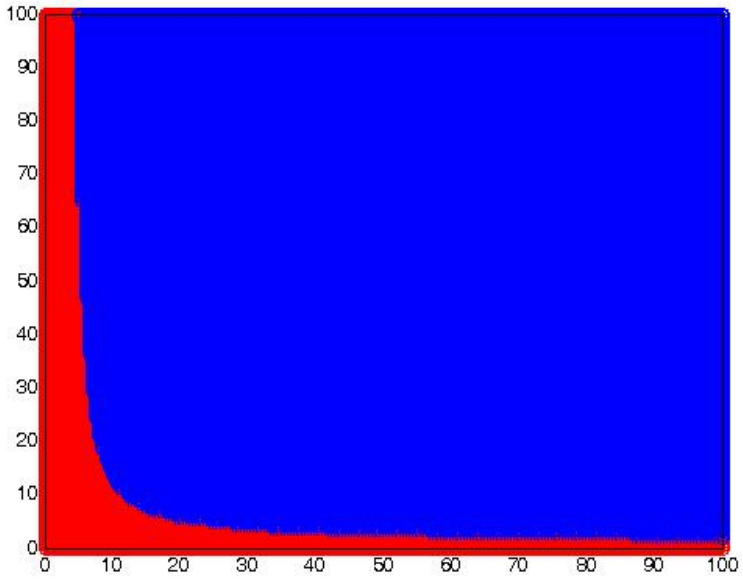

Figure 5.1: B=1.25

Conjecture 1. *There exists a decreasing function $\psi(x)$ whose graph has $x=\frac{1}{B-1}$ and $y=0$ as asymptotes. The points which lie below this curve are in the basin of attraction for the origin while for initial conditions above the graph of ψ the y_n components of the orbit will go to infinity, while the x_n components will approach the line $x = \frac{1}{B-1}$.*

In figures 5.1 through 5.3 we have plotted graphs resulting from numerical experimentation with red regions indicating initial conditions in the basin of the origin and blue regions indicating basins of ∞ for values of B=1.25, 1.333, and 1.5 respectively. From these we can get the basic outline of the function ψ conjectured.

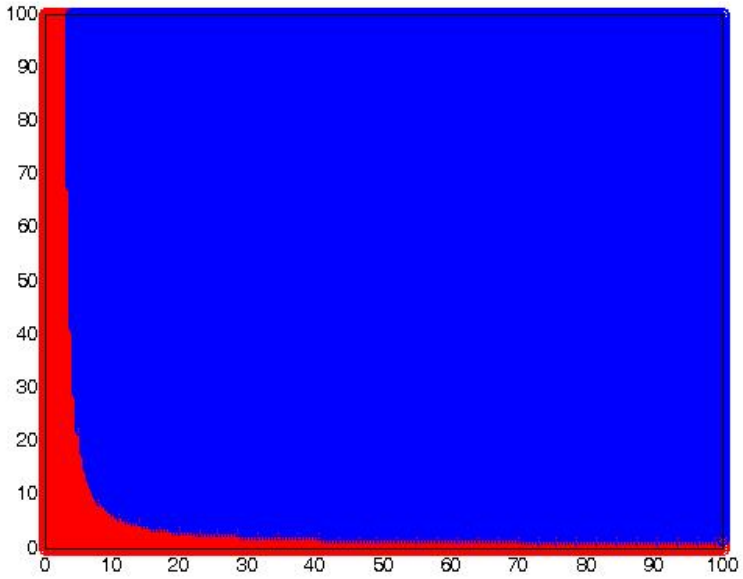

Figure 5.2: B=1.333

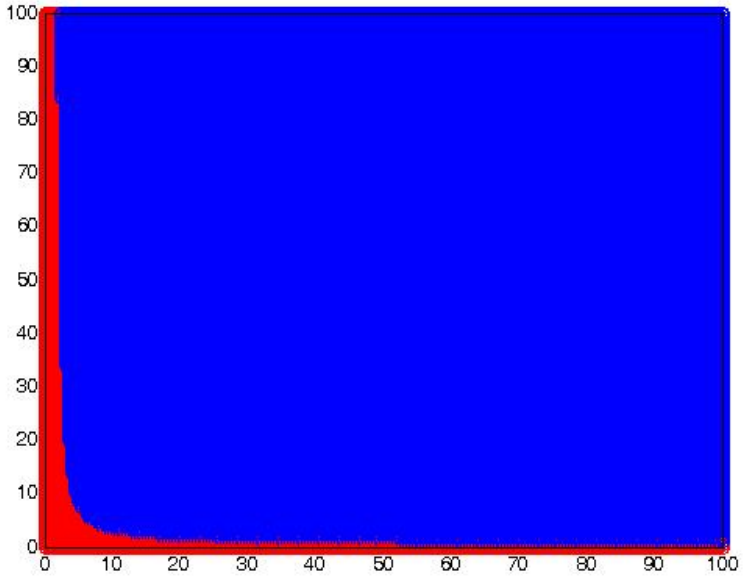

Figure 5.3: B=1.5

5.3 Example: A Difference-Delay Model of Flour Beetle Population Growth

In addition to modeling interactions between species, discrete population models can be used to model the intra-species dynamics when the biological life cycle of the species creates distinct forms of the organism. Such is the case for many insects and one of the preeminent examples in the literature is the larva-pupa-adult or LPA model. In the paper by Kuang and Cushing [15] the specific model of interest is given by the transition equations

$$L_{n+1} = bA_n e^{-c_{ea}A_n - c_{el}L_n}$$

$$P_{n+1} = (1-\mu_l)L_n$$

$$A_{n+1} = P_n e^{-c_{pa}A_n} + (1-\mu_a)A_n$$

where L, P, A are the population variables representing the number of Larvae, Pupae and Adult flour beetles, each time step represents the time it take for a larva to mature to a pupa, b is the number of eggs an adult is expected to lay, and μ_l and μ_a are the probabilities of non-cannibalism related deaths for a larva or adult respectively. Since the model is derived from a species of flour beetle known to be cannibalistic, the exponential factors represent probabilities of survival in the presences of other flour beetles and the respective "c" coefficients, c_{ea}, c_{el} and c_{pa} are called the cannibalism coefficients which parameterize these rates.

We can use the above system to write the equation indicating the number of adults at time n as a delay equation only in terms of A as

$$A_{n+1} = (1-\mu_a)A_n + b(1-\mu_l)A_{n-2}e^{-c_{ea}A_{n-2} - c_{pa}A_n}$$

This governs the system for cases of $n \geq 2$, while we can determine the necessary initial

conditions (A_0, A_1, A_2) from the (A_0, L_0, P_0) initial conditions by noting that

$$A_1 = P_0 e^{-c_{pa} A_0} + (1 - \mu_a) A_0$$

and

$$A_2 = (1 - \mu_l) L_0 e^{-c_{pa} A_1} + (1 - \mu_a) A_1$$

In the context of a population model it only makes sense to assume that all initial conditions are non-negative, and under such conditions Kuang and Cushing derived some global asymptotic stability results for the unique positive equilibrium of the above system when it exists by application of the following theorem from Hautus and Bolis[12].

Theorem 5. *Consider the delay difference equation given by*

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}) \qquad n \geq 0$$

with F a continuous function on the region $D \subseteq R^n$ which is increasing in each of its arguments. Let $\bar{x}$ be an equilibrium of the system defined by this equation and I be an interval containing $\bar{x}$ such that hypercube I^{k+1} is contained in D. Assume that for $u \neq \bar{x}$ we have

$$(u - \bar{x}) F(u, u, ..., u) < 0$$

then for initial conditions in I we have x_n is in I for all n and

$$\lim_{n \to \infty} = \bar{x}.$$

We begin by re-parameterizing the system and letting $\alpha = 1 - \mu_a$, $\beta = b(1 - \mu_l)$, $c_1 = c_{ea}, c_2 = c_{pa}$ and $x_n = A_{n+2}$, we rewrite the above delay equation as

$$x_{n+1} = \alpha x_n + \beta x_{n-2} e^{-c_1 x_{n-2} - c_2 x_n}.$$

Working with the reasonable assumptions that all initial conditions are positive, $c_1 + c_2 > 0$, and noting that the probabilistic nature of the μ's is such that $\alpha > 0$ and $\beta > 0$, we can see that $\alpha + \beta \leq 1$ implies the origin is the only equilibrium. While if $\alpha + \beta > 1$ there will be another positive equilibrium at $\bar{x} = \frac{1}{c_1+c_2} \ln\left(\frac{\beta}{1-\alpha}\right)$. The linearized system about an $\hat{x}$ in the new parameters is given by

$$x_{n+1} = (\alpha - \beta c_2 \hat{x} e^{-(c_1+c_2)\hat{x}})x_n + \beta(1 - c_1\hat{x})e^{-(c_1+c_2)\hat{x}}x_{n-2},$$

about the origin this is equivalent to the system

$$x_{n+1} = \alpha x_n + \beta x_{n-2}$$

and is asymptotically stable if and only if $\alpha + \beta < 1$ while, letting

$$A = \frac{c_2(1-\alpha)}{c_1 + c_2} \ln\left(\frac{\beta}{1-\alpha}\right) - \alpha$$

and

$$B = (1-\alpha)\left[\frac{c_1}{c_1 + c_2} \ln\left(\frac{\beta}{1-\alpha}\right) - 1\right],$$

the linearized system about the $\bar{x}$ equilibrium is, equivalent to the system

$$y_{n+1} = -Ay_n - By_{n-2}.$$

Kocic and Ladas have shown that the equilibrium $\bar{x}$ is stable if and only if $|B + A| < 1, |A - 3B| < 3$, and $B(B - A) < 1$ [14]. Using these results, Kuang and Cushing have determined global properties of such systems, notably if $\alpha + \beta \leq 1$ the origin is globally asymptotically stable.

To show this, it should be noted that given a positive initial value, orbits generated by the system will have positive terms ad infinitum, since for any index $k \geq n$ inspection of our

delay-formed equations makes it obvious that $x_k \geq \alpha^{k-n} x_n > 0$. Proceeding, if for a given n we take $\bar{x_n}$ to be the max of our three most recent values: x_n, x_{n-1}, x_{n-2} then we have for all solutions of the system

$$x_{n+1} \leq \alpha \bar{x_n} + \beta \bar{x_n} e^{-c_1 x_{n-2} + c_2 x_n}$$

Using the positivity restrictions on our parameters, we can see then that

$$x_{n+1} \leq \alpha \bar{x_n} + \beta \bar{x_n} \leq \bar{x_n}$$

and we deduce that the orbit of the sequence $\{\bar{x_n}\}$ is non-increasing. If we take the limit, $\bar{x}$, of this positive sequence to be greater than zero, then we know that there exists a natural number N such that $n > N$ implies that $\bar{x_n} < \bar{x} + \epsilon$ for any $\epsilon > 0$. But then for n greater than $N + 2$, (using the fact that $x_k \geq \alpha^{k-n} x_n$ for $k \geq n$) we know that $x_n \geq \alpha^2 \bar{x_n} \geq \alpha^2 \bar{x}$ so

$$x_{n+1} < (\bar{x} + \epsilon)(\alpha + \beta e^{-(c_1 + c_2)\alpha^2 \bar{x}}) < \bar{x} - \epsilon$$

which contradicts $\bar{x}$ being the limit of the non-increasing sequence. Thus the limit of the orbit must be zero.

Now in the case where $\alpha + \beta > 1$ there will be another, positive equilibrium. Kuang and Cushing have shown that in this case if $\beta < \min(e(1-\alpha), \frac{e\alpha c_1}{c_2})$ then this emergent equilibrium is globally asymptotically stable, provided that our sequence's initial term is positive.

Their proof requires the following facts: if $\alpha + \beta > 1$ then

$$\limsup x_n \leq \frac{\beta}{c_1 e(1-\alpha)}$$

and if we let

$$F(x_n, x_{n-1}, x_{n-2}) = \alpha x_n + \beta x_{n-2} e^{-c_1 x_{n-2} - c_2 x_n},$$

then for $u > 0$ and not equal to the non-zero equilibrium $\bar{x}$

$$(u - \bar{x})(F(u, u, u) - u) < 0$$

If we take the partial derivatives with respect to each variable we see that

$$\frac{\partial F}{\partial x_n} = \alpha - c_2 x_{n-2} \beta e^{-c_1 x_{n-2} - c_2 x_n},$$

$$\frac{\partial F}{\partial x_{n-1}} = 0,$$

and

$$\frac{\partial F}{\partial x_{n-2}} = \beta(1 - c_1 x_{n-2}) e^{-c_1 x_{n-2} - c_2 x_n}.$$

Using these facts Kuang and Cushing have shown there exists a positive N such that $n > N$ implies that $x_{n-2} < \frac{1}{c_1}$. Consider the interval $I = (0, \frac{1}{c_1})$ and the cube I^3, by the non-increasing property of the solution orbit we know that the non-zero equilibrium is in I^3, while $\frac{\partial F}{\partial x_{n-1}} = 0$ and $\frac{\partial F}{\partial x_{n-2}} > 0$. Since x_{n-2} is assumed to be non-negative we have

$$x_{n-2} e^{-c_1 x_{n-2} - c_2 x_n} \leq \frac{1}{e c_1}.$$

Using the partial derivatives we've established we can now see that

$$\frac{\partial F}{\partial x_n} \geq \alpha - \frac{c_2 \beta}{c_1 e} \geq 0.$$

Thus F is increasing in all of its arguments in I^3 and from the results of Hautus and Bolis we can conclude that the non-origin equilibrium is globally asymptotically stable.

Reminding ourselves that our model concerns cannibalistic flour beetles and that our

parameter β was a function of the larval survival probabilities and adult reproduction rates we can see that as these vary we go from a state of forced extinction to a positive equilibrium for population, which at least intuitively is quite plausible in the biological situation.

Chapter 6

Applications to Economics

Though from above we saw that the genesis of discrete dynamical systems appears to lie in its application to biology by Fibonacci in the 13th century, the approach also lends itself to modeling situations which arise in the social sciences. After all much of Fibonacci's famed "Liber Abaci" concerned business and economic calculations. In particular, in economic systems, the state of the system changes in discrete time steps as agents cannot allocate resources or make transactions in continuous time. Of course, economic system models have their own difficulties which distinguish them from biological or physical ones. One must take into account the possible indeterminacy in human behavior, and make assumptions regarding people's decision making which may prove to be inaccurate. Normally, this involves assumptions of a game theoretic nature: that agents are rational in a certain sense, that they have perfect information regarding the system's current state and conditional future trajectory, etc. But these seemingly unrealistic and restrictive assumptions are necessary in order to obtain any tractable mathematical model and with careful consideration and empirical investigation one can still hope to obtain useful results.

In this section we will investigate two applications of discrete dynamical systems to economic modeling. The first, involving Cournot Duopoly, in such a case there are only two producers in competition, and results come from assumptions about the economic agents

actions and the structure of supply and demand in the market. The second case, dealing with markets where agents display multiple strategies, depends on typical behavior patterns and uses rationality assumptions to determine the proportions of differing types of behavior among market participants.

Though the first subject decidedly falls into the category of "micro" economics, the latter operates in the gray area between "micro" and "macro" phenomena. It is my own hope that further investigations into discrete nonlinear dynamics will help to bridge the gap between the micro and macro aspects of economics in a general theory which will better explain both.

6.1 Cournot Duopoly: Stability of the Cournot Point and its Bifurcation Dynamics

Our first example deals with competition between producers when the market for their products cannot be said to perfectly competitive in the sense that an ideal commodity market is. In the standard case of a commodity market, many producers compete amongst each other in an open market to sell to consumers a uniform, substitutable product. In such cases the standard theory says that the prices that the producers must charge, and which the consumers must pay, is a function of the aggregate supply of and demand for the product; any given individual producer's or consumer's preferences or decisions have a negligible effect on the market price of the product. This is an accurate description of how markets function where there are many sellers and when different producers products are seen as interchangeable by the consumers and thus no advantage is to be gained from branding or other differentiation, *and* when there are enough active consumers that no individual or like-minded group of consumers can conspire to bid down the price to a significant extent. Such a case is nearly obtained for most wholesale food products, gasoline,and raw materials like copper, lumber, etc.

An alternative situation arises when high barriers to entry (or, in more nefarious situ-

ations, conspiracy) allow a single producer to dominate a market. In this case the single producer is said to have monopoly (i.e. single seller) power. From its earliest days as an academic discipline, economist have warned of the poor outcomes which arise when a firm obtains this kind of power.

The situation which we are interested in is one where neither of these extremes of perfect competition nor monopoly power are obtained. In such a case a small number of firms compete in a market for a large number of customers with a fixed demand function. The price of the good is then heavily dependent on the choices of each of the individual firms. Such a case is called an oligopoly and the instances of it in our society are myriad: smart phones, beverages, trucks, turbines, box cereal, etc. When choosing how much to produce and sell each firm must take not only what other firms will choose into account, but how other firms will react to their choice, and how other firms anticipate how each other will react, etc. This creates a game theoretical dynamic which requires a great deal more effort to parse and can lead to anything form steady equilibrium to chaotic and unstable behavior.

In the literature such systems are referred to as "Cournot Oligopolies", named for the French mathematician and political economist Antonie Augustin Cournot who first studied them in the early nineteenth century.

We will present some results from Puu [22] when there are two firms, Firm A and Firm B, in market competition, i.e. a "duopoly". If A brings x units of product to market at marginal cost α, while B brings y units to market at marginal cost β we assume the price commanded, p, is inversely proportional to supply so that

$$p = \frac{1}{x+y}.$$

The profits of the firms, P_A and P_B, respectively, will be given by

$$P_A = \frac{x}{x+y} - \alpha x,$$

and

$$P_B = \frac{y}{x+y} - \beta y.$$

If each firm takes the output of the other firm as given (i.e. constant) they can use standard calculus techniques to maximize their profit function. In the case of firm A they would note that

$$\frac{\partial P_A}{\partial x} = \frac{y}{(x+y)^2} - \alpha.$$

Setting this equal to zero they would get

$$(x+y)^2 = \frac{y}{\alpha},$$

thus

$$x = \sqrt{\frac{y}{\alpha}} - y.$$

This implies that the decision for how much firm A should bring to market is entirely dependent on their costs and how much firm B brings to market. The same, symmetrical reasoning applies to firm B, so

$$y = \sqrt{\frac{x}{\beta}} - x$$

If we consider the the two profit maximization curves simultaneously their intersection is what is called the "Cournot Point". At this point the profit commanded by firm A will be $\frac{\beta^2}{(\alpha+\beta)^2}$ while the profit of firm B will be $\frac{\alpha^2}{(\alpha+\beta)^2}$.

This assumes a static system where both agents know perfectly what the other agent will produce and this knowledge has no corresponding effect on the other agent's decisions. This is very unrealistic. However, it is plausible to assume that the system is evolving in discrete time steps and that each firm knows how much the other firm brought to market in the previous time period, and will use that to determine how much to bring to market in this time period. So that the sequence of the quantity of goods each firm brings to market

$\{(x_n, y_n)\}$ is governed by the system of equations

$$x_{n+1} = \sqrt{\frac{y_n}{\alpha}} - y_n$$

$$y_{n+1} = \sqrt{\frac{x_n}{\beta}} - x_n$$

with suitable restrictions on the admissible values of parameters and variables.

The Cournot point is an equilibrium for the iterative system, and, following standard practice, we would like to know about the stability and potential for bifurcations that may occur at this equilibrium. The Jacobian for the above system is

$$\begin{bmatrix} 0 & \frac{1}{2\sqrt{\alpha y_n}} - 1 \\ \frac{1}{2\sqrt{\beta x_n}} - 1 & 0 \end{bmatrix}$$

The eigenvalues of which at the Cournot point are $\lambda = \pm\frac{\beta-\alpha}{4\alpha\beta}$ Thus the stability of the Cournot point depends on the relative difference in the producers marginal costs, α and β. If we assume a case where the marginal costs start out close together and then diverge, as $|\frac{\beta-\alpha}{4\alpha\beta}|$ passes crosses the unit circle theoretical considerations previously examined tell us that the Cournot point will lose its stability and chaotic period doubling dynamics occur. This sheds interesting light on the oligopolistic situation being models, as the ratio of the marginal costs for the individual firms change its possible for chaotic behavior to take hold in the price of the product. This runs counter to the naive theory of market behavior, which assumes many producers, that price should always tend towards equilibrium. In the Cournot model the usual expectations are fulfilled when the ratio of the firms' marginal costs is within a prescribed range (which happens to be $\frac{\beta}{\alpha} \in (3-\sqrt{2}, 3+\sqrt{2})$ or $\frac{\alpha}{\beta} \in (3-\sqrt{2}, 3+\sqrt{2})$). But as the ratio exits this interval we observe wild oscillations in price behavior as opposed to the plausible assumption that the firm with lower marginal costs drives its competitor from the marketplace. The result lends to the plausibility to the claim that the market system

has essential tendencies to instability when oligopolistic dynamics are considered. This is an example of the period doubling bifurcation leading to chaotic behavior. Other bifurcations are possible though when further considerations are brought into play. The system given by the transitions equations

$$x_{n+1} = \sqrt{\frac{y_n}{\alpha}} - y_n$$

$$y_{n+1} = \sqrt{\frac{x_n}{\beta}} - x_n,$$

can be decoupled into two autonomous systems given by the cumbersome equations:

$$x_{n+2} = \sqrt{\frac{\sqrt{\frac{x_n}{\beta}} - x_n}{\alpha}} - \sqrt{\frac{x_n}{\beta}} + x_n$$

$$y_{n+2} = \sqrt{\frac{\sqrt{\frac{y_n}{\alpha}} - y_n}{\beta}} - \sqrt{\frac{y_n}{\alpha}} + y_n.$$

But other dynamics can arise when we consider situations which don't lead to decoupling in such a manner. Such a case arises when we consider each agents decision to lag by a certain degree, which can be modeled by introducing "adjustment speed" factors, say c for firm A and d for firm B, each between 0 and 1. Then the governing equations of the system become:

$$x_{n+1} = (1-c)x_n + c\left(\sqrt{\frac{y_n}{\alpha}} - y_n\right)$$

$$y_{n+1} = (1-d)y_n + d\left(\sqrt{\frac{x_n}{\beta}} - x_n\right)$$

In the extreme case of $c, d = 1$ we are back in the case already discussed, in the system obtains the alternate extreme, $c, d = 0$, we are in the case where $x_{n+1} = x_n$ and $y_{n+1} = y_n$ so that the system is static for all values on n. It happens that in this case the equilibria of the system remain the origin and the Cournot point, however the stability dynamics of the Cournot point relative to our new parameters manifest new behavior. If we investigate the

Jacobian of the new system we see that is in the form:

$$\begin{bmatrix} 1-c & c\left(\frac{1}{2\sqrt{\alpha y_n}}-1\right) \\ d\left(\frac{1}{2\sqrt{\beta x_n}}-1\right) & 1-d \end{bmatrix}.$$

Again it is obvious that in the case where both of our new lag parameters equal 1 we are back in the original situation. At the Cournot point this evaluates to

$$\begin{bmatrix} 1-c & c\left(\frac{\beta-\alpha}{2\alpha}\right) \\ d\left(\frac{\alpha-\beta}{2\beta}\right) & 1-d \end{bmatrix}.$$

The characteristic equation of this matrix is

$$\lambda^2-(2-c-d)\lambda+1-(c+d)+cd(1+\frac{(\alpha-\beta)^2}{4\alpha\beta}).$$

We can conclude that our eigenvalues will have the form

$$\lambda=\frac{2-c-d\pm\sqrt{c^2+d^2-cd(\frac{\alpha^2+\beta^2}{4\alpha\beta})}}{2}$$

If it is the case that both of the eigenvalues are less than 1 in modulus we will have a stable equilibrium and we examine what happens when either eigenvalue passes through this critical value. Since the determinant of a matrix is the product of its eigenvalues we can infer that a change in stability will occur at the Cournot point when the determinant goes from being less than 1 in modulus to greater than 1 in modulus. Algebraically this occurs when

$$(1-c)(1-d)+cd\frac{(\alpha+\beta)^2}{4\alpha\beta}=1,$$

that is to say

$$c+d=cd\left(1+\frac{(\alpha+\beta)^2}{4\alpha\beta}\right),$$

which leads to the conclusion

$$4\alpha\beta(\frac{1}{c}+\frac{1}{d}-1)=(\alpha-\beta)^2.$$

We can substitute this into the discriminant of our parameterized eigenvalues to determine via algebriac manipulation that

$$\lambda=\frac{2-c-d\pm\sqrt{(c+d)(c+d-4))}}{2}.$$

Both c and $d$are restricted to values between 0 and 1, so this says that the eigenvalues of the Jacobian at the Cournot point must be complex as the equilibrium loses stability, thus we can conclude that a Neimark-Sacker bifurcation has then occurred generating a closed, invariant curve. In this case the new curve is "sub-critical" meaning that it is an unstable set. Understanding such bifurcations requires exceptionally subtle analysis and readers are encouraged to see [10] and [22] for more details on this subject.

6.2 Markets with Heterogeneous Agents: Stability and Bifurcation Dynamics due to "Intensity of Choice"

In the above case the prevailing price of the objects sold depended on the game theoretic dynamics between both producers. In a perfectly competitive market, i.e. one with many producers competing as opposed to only two, the price commanded for a unit produced will depend on the aggregate demand for the product and on how much quantity the producers bring to market. But how much each producer brings to market is in turn determined by the price which the product is expected to fetch. Thus, assuming a constant aggregate demand, the price commanded is a function of the price expected, and the price which is expected

in the next time step usually is some function of the price commanded in the current or in a previous chain of time steps. This feedback dynamic creates a great deal conceptual difficulty when one wants to try to model the situation that arises in a perfectly competitive market.

One way of simplifying the situation and to make it amiable to mathematical modeling which arose during the 1970's is the rational expectations hypothesis or REH. Championed by the "freshwater" school of economics and notably by the Nobel Laureate Robert Lucas, under the rational expectations hypothesis all agents are assumed to have access to unbiased statistics that accurately predict the future values of prices, thus, except for statistical fluctuations due to chance, the expected price at time $n+1$ will be the actual price at time $n+1$. Thus prices reflect all the information available plus or minus some statistical error.

By assuming this degree of similarity amongst the agents in both strategy and in knowledge, markets can be assumed to be efficient in that no slack utility is left on the table once transactions have all cleared. This deduction from the rational expectations hypothesis is called the efficient market hypothesis or EMH [13]. For anyone who has interacted with human being in business or in financial markets such assumptions stretch one's credulity. Economists have noted the seeming unrealistic-ness of the modeling assumptions and have offered alternative descriptions. The one of most interest to us here is that of heterogeneous agents, i.e. the assumption that market participants have diverse expectation strategies and knowledge levels as opposed to uniform rational expectations and perfect knowledge. Brock and Hommes ([2],[3],[13] have considered such models and the following is a simplified version of their presentation. We assume that the demand function, D, for a given product is decreasing linearly in price while the supply function, S, is increasing linearly in price so that

$$D = d - \alpha p$$

$$S = \beta p$$

with α, β, d positive real constants. Now we assume there are two types of firms competing in the market, type A firms are "naive" or trend-chasers and simply expect that the previous time period's price will be the current time period's price while type B firms are "sophisticated" and have perfect knowledge of what the price will be in each time period. Thus the forecasting rule, i.e. the expected price at time t denoted $H(p_t)$ for type A firms is $H(p_t) = p_{t-1}$ while for type B firms this rule will be $H(p_t) = p_t$. The market equilibrium, that is the point where supply equals demand will then be

$$d - \alpha p_t = \beta(\gamma p_{t-1} + (1-\gamma)p_t)$$

where $\gamma \in [0, 1]$ is the proportion of the market participants adopting the naive strategy. we can solve for p_t obtaining:

$$p_t = \frac{\alpha - \gamma\beta p_{t-1}}{d + (1-\gamma)\beta}.$$

Of course in the real situation obtaining such perfect information as type B firms obtain has costs, while the simplistic rule used by type A firms incurs no cost. Also we should consider the fact that the greater proportion of type B firms in the market (i.e. the closer γ is to zero) then the more and more p_t approaches the constant values $\frac{\alpha}{d+\beta}$, thus the naive rule of a type A firm will return an almost identical value prediction as the rule determined by the sophisticated and costly information type B firms obtain, thus it would be pointless to incur the extra cost of independently obtaining perfect information when following the crowd gives identical results. Also as $\gamma \to 1$ the difference between the expected values of the type A's and the actual values (which the type B's obtain) increase, which given a reasonable cost for information could make it a superior strategy to be a type B firm.

Thus, if we assume that firms have the ability to switch their type from A to B or vice versa, and thus vary the γ parameter, different strategies will be optimal depending on the current and expected composition of the market. If we assume that most agents are bounded rational agents, that is they prefer to use the strategy which, costs considered, maximizes

their profit, the degree to which agents are willing to jump from the less effective to the more effective strategy is called the "intensity of choice" and clearly the parameter γ, and thus the systems behavior as a whole, is dependent on this intensity.

For example we can consider a market system where there is a single stock for sale, the stock is assumed to have a fundamental underlying price, p_t, though in any given period the actual price will be the fundamental price plus some deviation σ_t, we will again have two types of firms or agents, those who think that the price at the next period will be the fundamental price, i.e. that $\sigma_{t+1} = 0$ so that $p_{t+1} = f_{t+1}$ where f_{t+1} is the fundamental price at time t, and those that think that the price in the next period will be subject to deviation in the same direction it was subject to previously $p_{t+1} = g\sigma_t + f_{t+1}$ where g is a positive constant trend parameter. If g is smaller than 1 plus the risk free rate of return (hereafter denoted R, what an investor could get by parking their money in US Treasuries) then the resulting system is always stable (see [12]) for g much larger than R,i.e. bigger than R^2, the usual price equilibrium is globally unstable. If $R < g < R^2$ then the stability of the price equilibrium will depend on the intensity of choice, with stability following from low intensity, and instability from high intensity. Additionally a pitchfork bifurcation may occur, leading to two additional steady states of the system. Increases in the intensity of choice can also lead to chaotic behavior, providing what Brock and Hommes [2] call a "rational route to randomness" which can help to explain the apparent stochastic behavior of prices in financial markets while maintaining the rational-agents based modeling prospective, due to the nonlinearities involved in the system.

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Appendix A

Code List

```
%Thesis Simulation 3.3
A=10;
B=5;
x(1)=2;
x(2)=1;
for i=2:100
    x(i+1)=A*x(i)/(1+B*x(i-1));
end
plot(x,'b*')
line('xdata',[0,105],'ydata',[9/5,9/5])

%Thesis Simulation 5.2
%B=1.25;
%B=1.3333
%B=1.5
Initialx=[0:0.5:100];
Initialy=[0:0.5:100];
```

```
for i=1:length(Initialx)
    for k=1:length(Initialy)
        clear x
        clear y
        x(1)=Initialx(i);
        y(1)=Initialy(k);
        for j=1:99
            x(j+1)=x(j)*y(j)/(1+y(j));
            y(j+1)=B*y(j)*x(j)/(1+x(j));
        end
            if x(100)<0.001
               plot(x(1),y(1), 'ro')
                hold on
            else
                plot(x(1),y(1), 'bo')
            end

    end
end
```